On OBDD based algorithms and proof systems that dynamically change order of variables

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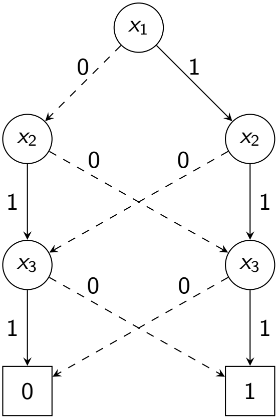
Ordered binary decision diagram (OBDD)



- OBDD represents Boolean function $\{0,1\}^n \to \{0,1\}$
- π is order of variables; if $i < j$, then $x_{\pi(j)}$ can't appear before $x_{\pi(i)}$.

Operations with OBDD

Given	Compute	Complexity
D^π	whether D^π is satisfiable	$O(\|D\|)$
D^π	$(\neg D)^\pi$	$O(\|D\|)$
D_1^π, D_2^π	$(D_1 \wedge D_2)^\pi$	$O(\|D_1\| \times \|D_2\|)$
D_1^π, D_2^π	$(D_1 \vee D_2)^\pi$	$O(\|D_1\| \times \|D_2\|)$
D^π, x is a variable	$(\exists x D)^\pi$	$O(\|D\|)$
D^π, ρ	$(D\|_\rho)^\pi$	$O(\|D\|)$
$D_1^{\pi_1}, \pi_2$	$D_2^{\pi_2}$ such that $D_1^{\pi_1} \equiv D_2^{\pi_2}$	$poly(\|D_1\| \times \|D_2\|)$
D^π	min D_0^π such that $D_0 \equiv D$	$O(\|D\|)$
$D_1^{\pi_1}, D_2^{\pi_2}, D_3^{\pi_1}$	whether $D_3 \equiv D_1 \wedge D_2$	NP-hard

$D_1 \equiv D_2$ if D_1 and D_2 represents the same Boolean function.

Outline

- OBDD($\wedge$, weakening)-proof system
- Lower bounds for OBDD($\wedge$, reorder)-proof system
- Lower bounds for OBDD($\wedge, \exists$, reorder)-algorithms

OBDD(∧, weakening)-proofs

- $\phi = C_1 \wedge C_2 \wedge \cdots \wedge C_t$ is unsatisfiable CNF.
- Choose order π; every C_i is represented as π-ordered OBDD.
- Join (conjunction) rule: $\frac{D_1^\pi, D_2^\pi}{(D_1 \wedge D_2)^\pi}$
- Weakening rule: $\frac{D^\pi}{D_1^\pi}$ if $D \models D_1$.
- Proof of unsatisfiability of ϕ: derivation a constant false OBDD.
- [Atserias, Kolaitis, Vardi, 2004] OBDD(∧, weakening) simulates CP* $\implies$ PHP_n^{n+1} has proofs of poly size.
- OBDD(∧, weakening) is stronger than Resolution
- Unsatisfiable linear systems over GF(2) have short proofs
- [Segerlind, 2007] $2^{n^{\Omega(1)}}$-lower bound for tree-like OBDD(∧, weakening)-proofs
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OBDD($\wedge$)-proofs

- [Groote, Zantema, 2003; Tveretina et al., 2009] OBDD($\wedge$)-proof system is incomparable with resolution
- [Tveretina et al., 2009] PHP_n^{n+1} requires OBDD($\wedge$)-proofs of size $2^{\Omega(n)}$
- [Friedman, Xu, 2013] Random 3CNFs are hard for OBDD($\wedge$)-proofs in two particular cases:
  - with a fixed order of the variables
  - with fixed orders of application of rules

Reordering rule

- Reordering rule: $\frac{D_1^{\pi_1}}{D_2^{\pi_2}}$ if $D_1^{\pi_1} \equiv D_2^{\pi_2}$
- Join (conjunction) rule: $\frac{D_1^{\pi}, D_2^{\pi}}{(D_1 \wedge D_2)^{\pi}}$
- OBDD($\wedge$, reorder)-proof system:
  - We exponentially separate OBDD($\wedge$, reorder) from OBDD($\wedge$)
  - Lower bound $2^{\Omega(n)}$ for PHP_n^{n+1}.
  - Lower bound $2^{\Omega(n)}$ for Tseitin formulas.

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Lower bound method for OBDD($\wedge$, reorder)

Let $\Phi = \bigwedge_{i \in I} C_i$ be minimally unsatisfiable CNF

1. Φ' is a satisfiable formula associated with Φ. Roughly speaking: Φ' is Φ without several clauses.
   - For Φ=PHP_n^{n+1}, Φ'=PHP_n^n
   - For unsatisfiable Tseitin formulas, Φ' is satisfiable Tseitin formula.
2. Prove that any OBDD representation of Φ' has large size.
3. The last step: $\frac{F_1^\pi \wedge F_2^\pi}{0}$. F_1, F_2 are satisfiable and $F_1 \equiv \bigwedge_{i \in I_1} C_i, F_2 \equiv \bigwedge_{i \in I_2} C_i$ and $I_1 \neq I_2$, $I_1 \cup I_2 = I$.
4. Find partial substitution ρ_1, ρ_2 with same support: $F_1|_{\rho_1} \wedge F_2|_{\rho_2}$ is a hard satisfiable formula for OBDD. Hence either $F_1|_\rho^\pi$ or $F_2|_\rho^\pi$ is hard for OBDD, hence F_1 or F_2 is hard for OBDD.

Lower bounds for OBDD

For particular order π:

- F, $S = \{x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(\ell)}\}$.
- Let $\rho_1, \rho_2, \dots, \rho_k$ be partial substitution with support S such that $F|_{\rho_1}, F|_{\rho_2}, \dots, F|_{\rho_k}$ are different functions.
- Then every π-OBDD for F has at least k vertices.

For all orders:

- For arbitrary S that consists of ℓ variables.

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Tseitin formulas

- $G(V, E)$ is undirected constant-degree graph;
- For every $e \in E$: x_e Boolean variable;
- $c : V \to \{0, 1\}$ labelling function
- $TS_{G,c} = \bigwedge_{v \in v} \left(\bigoplus_{u:(u,v) \in E} x_{(u,v)} = c(v) \right)$

Lemma. $TS_{G,c}$ is satisfiable iff for every connected component U, $\bigoplus_{v \in U} c(v) = 0$.

OBDD for satisfiable Tseitin formula

- Let $TS_{G,c}$ be satisfiable Tseitin formula.
- Consider some order π;
- Let S be a set that consists first ℓ edges according π
- Consider some substitution ρ with support S.
- $TS_{G,c}|_\rho = TS_{G',c+f}$, where $G'(V, E \setminus S)$ and $f : V \to \{0,1\}$ is a modification of labels made by ρ.
- Different functions: different f and $TS_{G',c+f}$ is satisfiable.
- We estimate the number of f such that
  - $TS_{G',c+f}$ is satisfiable
  - f can be obtained by a substitution $\iff$ $TS_{G'',f}$ is satisfiable, where $G''(V, S)$.
- $\sharp G' + \sharp G''$ linear conditions on f.
- Hence number of different functions at least $2^{n-\sharp G'-\sharp G''}$

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Tseitin formulas on expanders

Theorem. If $G(V, E)$ is good enough expander with $|V| = n$ then $\exists \ell$: $\forall S \subseteq E$ if $|S| = \ell$ then $\sharp G'(V, E \setminus S) + \sharp G''(V, S) \leq (1 - \epsilon)n$.

Corollary. Every OBDD representation of satisfiable $TS_{G,c}$ has size at least $2^{\Omega(n)}$.

Corollary. If G differs from good enough expander by at most $o(n)$ edges, then OBDD representation of satisfiable $TS_{G,c}$ has size at least $2^{\Omega(n)}$.

Lemma. Good enough expander is connected and remains connected after deleting of any two vertices and edges from the shortest path between them.

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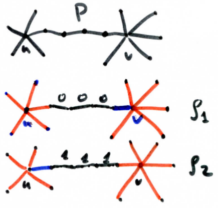
Lower bound for unsatisfiable Tseitin formula

Theorem. If $G(V,E)$ is good enough expander with $|V| = n$ and $TS_{G,c}$ is unsatisfiable then the size of any OBDD($\wedge$, reorder)-proof of $TS_{G,c}$ is at least $2^{\Omega}(n)$.

Proof.

- The last step: $\frac{F_1^\pi \wedge F_2^\pi}{0}$. F_1, F_2 are satisfiable.
- Let F_1 does not contains C_u and F_2 does not contain C_v and $(u,v) \notin E$
- Let P be the shortest uv-path



$F_1|_{\rho_1} \wedge F_2|_{\rho_2}$ is almost satisfiable $TS_{\tilde{G},c'}$, where $\tilde{G}(V \setminus \{u,v\}, E \setminus P)$.

OBDD($\wedge$, weakening, reorder)

Open questions:

- To separate OBDD($\wedge$, weakening, reorder)-proof system and OBDD($\wedge$, weakening)-proof system.
- Prove superpolynomial lower bound for OBDD($\wedge$, weakening, reorder)-proofs

Symbolic quantifier elimination

OBDD($\wedge, \exists$)-algorithms [Pan, Vardi, 2004] for SAT.
Input: CNF formula ϕ

1. Choose order π, D^{π}. Initially $D \equiv 1$.
2. $S := \{$clauses of $\phi\}$.
3. While $S \neq \emptyset$ apply the following operations:
   - Conjunction ($\wedge$) Choose $C \in S$; $S := S - C$; $D^{\pi} := D^{\pi} \wedge C$
   - Projection ($\exists$) If x does not appear in S, then $D^{\pi} := (\exists x D)^{\pi}$
4. If $S = \emptyset$ then report whether D is satisfiable or not.

Running time is polynomially connected with the size of the largest D.

OBDD($\wedge, \exists$)-algorithms

Upper bounds:

- [Chén, Zhang, 2009] Pigeonhole principle PHP_n^{n+1} is easy for OBDD($\wedge, \exists$)-algorithms.
- Tseitin formulas are easy for OBDD($\wedge, \exists$)-algorithms.
  - $\exists x \begin{cases} x + y + z = 1 \\ x + t + f = 0 \end{cases} \iff y + z + t + f = 1.$
  - Sum up all equalities in the connected component.

Lower bounds:

- Follows from lower bounds for (tree-like) OBDD($\wedge$, weakening)-proofs.

OBDD($\wedge, \exists$, reorder)-algorithms

- (reorder) Choose π' and $F^{\pi'}$ such that $F \equiv D$; $\pi := \pi'$ and $D := F$.

Our goals:

- Lower bounds for OBDD($\wedge, \exists$, reorder)-algorithms
- Lower bound of type $2^{\Omega(n)}$, where n is number of variables
- Lower bound for natural formulas.

Theorem. There is a randomized construction of a family of satisfiable formulas F_n on n variables in $O(1)$-CNF such that every OBDD($\wedge, \exists$, reorder)-algorithm runs at least $2^{\Omega(n)}$ steps on F_n. Formula F_n represents a system of linear equations over $\mathbb{F}_2$ based on checksum matrix of some linear code.

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OBDD size for codes

- A code $C \subseteq \{0,1\}^n$ recovers ρ fraction of erasures by a list of size L (or C is (ρ, L)-erasure list-decodable) if for all $w \in \{0,1,\square\}^n$ with at most ρ fraction of $\square$ there are at most L codewords that are consistent with w.

Lemma. If $C \subseteq \{0,1\}^n$ is $(\frac{1}{2}+\epsilon, L)$-erasure list-decodable, then any OBDD for χ_C has OBDD size at least $\frac{|C|}{L^2}$. Moreover for every $i_1, i_2, \ldots, i_k \in [n]$ if $k \le 2\epsilon n$, then $\exists x_{i_1} \ldots \exists x_{i_k} \chi_C$ has OBDD size at least $\frac{|C|}{L^2}$.

Proof.

- Consider some order π
- $\exists x_{n-k+1} \ldots \exists x_n \chi_C(x_1, x_2, \ldots, x_n)$
- We show that there are many substitutions to the first $\frac{n-k}{2}$ variables that produce different functions.
- S is a set of all $\frac{n-k}{2}$-size prefixes of C.
- $L+1$ different codewords can't have same prefixes since $n - \frac{n-k}{2} \le (\frac{1}{2}+\epsilon)n$. Hence $|S| \ge \frac{|C|}{L}$.

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Proof. (Continue)

- S is a set of all $\frac{n-k}{2}$-size prefixes of C. $|S| \geq \frac{|C|}{L}$.
- For every $s \in S$ we define ρ_s that substitutes $x_1 \ldots x_{(n-k)/2} := s$. $\exists x_{n-k+1} \ldots \exists x_n \chi_C(x_1, x_2, \ldots, x_n)|_{\rho_s}$ is satisfiable since s is a prefix of a codeword.
- Let $s_1, s_2, \ldots, s_{L+1}$ be different elements of S. We claim that ρ_{s_i} for $i \in [L+1]$ can't produce the same function.
- Let $s_1 r$ be a prefix of an element of C of size $n-k$.
- $\exists x_{n-k+1} \ldots \exists x_n \chi_C(x_1, x_2, \ldots, x_n)|_{\rho_{s_1}}(r) = 1$, hence for all $i \in [L+1]$, $s_i r$ is $n-k$ prefix of some element of C. Contradiction, since $\square^{\frac{n-k}{2}} r \square^k$ has at most $\frac{1}{2}+\epsilon$ fraction of $\square$.
- Number of different functions $\geq \frac{S}{L} \geq \frac{|C|}{L^2}$.

Lower bounds for OBDD($\wedge, \exists$, reorder)-algorithms

Theorem. Let A be an $0.97n \times n$ matrix over $\mathbb{F}_2$ such that

- A is a checksum matrix of $(\frac{2}{3}, 10)$-eras. list-decodable code;
- A contains $t = O(1)$ ones in every row;
- Every $\frac{n}{12}$ columns of A contain ones in at least $0.96n$ rows.

Then every OBDD($\wedge, \exists$, reorder)-algorithm runs at least $2^{\Omega(n)}$ steps on the formula that encodes $Ax = 0$.

Proof.

- Let D be the first diagram after $\frac{n}{12}$ applications of $\exists$.
- $D \equiv \exists_1 \dots \exists_{n/12} F$, where F is a conjunction of all clauses from $0.96n$ rows of A and possibly some other clauses.

Lemma. If A is a checksum matrix of (ρ, L)-erasure list-decodable code. A' is obtained from A by removing of k rows, then A is a checksum matrix of $(\rho, 2^k L)$-erasure list-decodable code.

- F is char. function of a $(\frac{2}{3}, 2^{0.1n}100)$-erasure list-decodable code of size at least $|C|$. Hence $|D| \geq 2^{\Omega(n)}$.

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Lemma. If A is a checksum matrix of (ρ, L)-erasure list-decodable code. A' is obtained from A by removing of k rows, then A is a checksum matrix of $(\rho, 2^k L)$-erasure list-decodable code.

- F is char. function of a $(\frac{2}{3}, 2^{0.1n}100)$-erasure list-decodable code of size at least $|C|$. Hence $|D| \geq 2^{\Omega(n)}$.

Lower bounds for OBDD($\wedge, \exists$, reorder)-algorithms

Theorem. Let A be an $0.97n \times n$ matrix over $\mathbb{F}_2$ such that

- A is a checksum matrix of $(\frac{2}{3}, 10)$-eras. list-decodable code;
- A contains $t = O(1)$ ones in every row;
- Every $\frac{n}{12}$ columns of A contain ones in at least $0.96n$ rows.

Then every OBDD($\wedge, \exists$, reorder)-algorithm runs at least $2^{\Omega(n)}$ steps on the formula that encodes $Ax = 0$.

Proof.

- Let D be the first diagram after $\frac{n}{12}$ applications of $\exists$.
- $D \equiv \exists_1 \ldots \exists_{n/12} F$, where F is a conjunction of all clauses from $0.96n$ rows of A and possibly some other clauses.

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Construction of code

Lemma. [Guruswami, 2003] If C is a code with relative distance δ, then for every $\epsilon > 0$ the code C is $((2-\epsilon)\delta, \frac{2}{\epsilon})$-erasure list-decodable.

[Gallager, 1962]

$$B = \left(\underbrace{\boxed{I_{n/t}}\boxed{I_{n/t}}\dots\boxed{I_{n/t}}}_{t \text{ times}} \right) \text{ is } n/t \times n \text{ matrix with } t \text{ ones per row.}$$

$$A = \begin{pmatrix} [\text{ 1st random perm. of columns of } B] \\ [\text{2nd random perm. of columns of } B] \\ \vdots \\ [r\text{-th random perm. of columns of } B] \end{pmatrix} \text{ is } \tfrac{rn}{t} \times n \text{ matrix}$$

with t ones per row.

Lemma. $\exists t$ such that for $r = 0.97t$ w.h.p A defines a code with relative distance 0.49. W.h.p every $\frac{n}{12}$ columns of A intersects at least $0.96n$ rows.

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Open problems

1. Lower bound for OBDD($\wedge$, weakening, reorder)-proofs;
2. Separate OBDD($\wedge$, weakening) and OBDD($\wedge$, weakening, reorder)-proofs;
3. Is it possible to simulate OBDD($\wedge$)-proofs by OBDD($\wedge, \exists$)-algorithms?
4. Prove lower bound for OBDD($\wedge, \exists$, reorder)-algorithms on unsatisfiable linear systems.
5. Compare OBDD($\wedge$)-proofs with constant degree Frege.