A Generalized Method for Proving Polynomial Calculus Degree Lower Bounds

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Workshop on Proof Complexity
Special Semester Program on Complexity Theory
St Petersburg, Russia
May 19, 2016

Joint work with Mladen Mikša

Proof Complexity and Expansion

- **General goal:** Prove that concrete proof systems cannot efficiently certify unsatisfiability of concrete CNF formulas

- **General theme:**

CNF formula F "expanding"

$$\Downarrow$$

Large proofs needed to refute F

- Well-developed machinery for resolution

- Very much less so for polynomial calculus

Main Message

Theorem (to be made precise later)

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- Extends an approach from [Alekhnovich & Razborov '01]
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- **Corollary:** Lower bound resolving problem in [Razborov '02]

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This talk:

- Theorem statement clean, but quite involved proof
- Therefore, present main ideas for resolution (way simpler)
- Bonus: general formulation of combinatorial conditions comparing and contrasting resolution and polynomial calculus

Outline

1. Proof Complexity Overview
   - Preliminaries
   - Resolution
   - Polynomial Calculus
2. Lower Bounds from Expansion
   - Resolution Width
   - Polynomial Calculus Degree
   - Pigeonhole Principle
3. Open Problems

Some Notation and Terminology

- Literal a: variable x or its negation $\overline{x}$
- Clause $C = a_1 \vee \cdots \vee a_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- CNF formula $F = C_1 \wedge \cdots \wedge C_m$: conjunction of clauses
- k-CNF formula: CNF formula with clauses of size $\leq k = \mathcal{O}(1)$
- M = size of formula = # literals ($\approx$ # clauses for k-CNF)
- N = # variables $\leq M$

The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (axioms)

Derive new clauses by resolution rule

$$\frac{C \lor x \qquad D \lor \overline{x}}{C \lor D}$$

Refutation ends when empty clause $\bot$ derived

1. $x \lor y$
2. $x \lor \overline{y} \lor z$
3. $\overline{x} \lor z$
4. $\overline{y} \lor \overline{z}$
5. $\overline{x} \lor \overline{z}$

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3.	$\overline{x} \lor z$	Axiom
4.	$\overline{y} \lor \overline{z}$	Axiom
5.	$\overline{x} \lor \overline{z}$	Axiom
6.	$x \lor \overline{y}$	Res$(2, 4)$
7.	x	Res$(1, 6)$
8.	$\overline{x}$	Res$(3, 5)$
9.	$\perp$	Res$(7, 8)$

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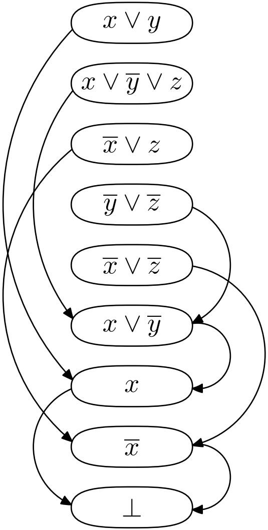

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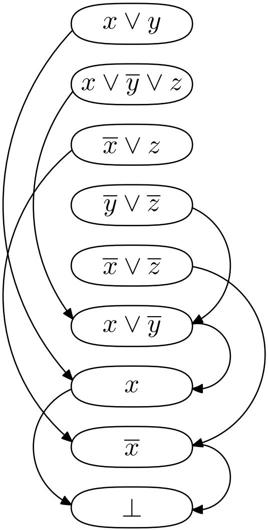
Tree-like resolution if DAG is tree



Resolution Size/Length

Size/length $=$ $\#$ clauses in refutation

Most fundamental measure in proof complexity

Never worse than $\exp(\mathcal{O}(N))$

Matching $\exp(\Omega(M))$ lower bounds known

(Recall $N = \#$ variables $\leq$ formula size $= M$)

Examples of Hard Formulas w.r.t Resolution Size (1/2)

Pigeonhole principle (PHP) [Haken '85]
"$n + 1$ pigeons don't fit into n holes"

Variables $p_{i,j}$ = "pigeon i goes into hole j"

$p_{i,1} \lor p_{i,2} \lor \cdots \lor p_{i,n}$	every pigeon i gets a hole
$\overline{p}_{i,j} \lor \overline{p}_{i',j}$	no hole j gets two pigeons $i \neq i'$

Can also add "functionality" and "onto" axioms

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But only lower bound $\exp\left(\Omega\left(\sqrt[3]{M}\right)\right)$ in terms of formula size

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Tseitin formulas [Urquhart '87]

"Sum of degrees of vertices in graph is even"

Variables = edges (in undirected graph of bounded degree)

- Label every vertex 0/1 so that sum of labels odd
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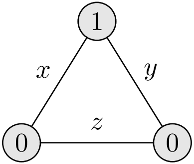

$$
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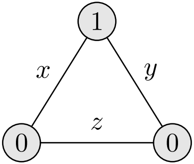

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Requires size $\exp(\Omega(M))$ on bounded-degree edge expanders
"Resolution cannot count $\bmod 2$**"**

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Proof: at most $(2N)^{\text{width}}$ distinct clauses
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Much less obvious. . .

Width Lower Bounds Imply Size Lower Bounds

Theorem ([Ben-Sasson & Wigderson '99])

For k-CNF formula over N variables

$$\textit{proof size} \geq \exp\left(\Omega\left(\frac{(\textit{proof width})^2}{N}\right)\right)$$

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For tree-like resolution have proof size $\geq 2^{\text{width}}$ [BW99]

General resolution: width up to $\mathcal{O}\big(\sqrt{N \log N}\big)$ implies no size lower bounds — possible to tighten analysis? **No!**

Optimality of the Size-Width Lower Bound

Ordering principles [Stålmarck '96, Bonet & Galesi '99]
"Every (partially) ordered set $\{e_1, \ldots, e_n\}$ has minimal element"

Variables $x_{i,j} =$ "$e_i < e_j$"

$\overline{x}_{i,j} \lor \overline{x}_{j,i}$	anti-symmetry; not both $e_i < e_j$ and $e_j < e_i$
$\overline{x}_{i,j} \lor \overline{x}_{j,k} \lor x_{i,k}$	transitivity; $e_i < e_j$ and $e_j < e_k$ implies $e_i < e_k$
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Refutable in resolution in size $\mathcal{O}\bigl(N^{3/2}\bigr) = \mathcal{O}(M)$
Requires resolution width $\Omega\bigl(\sqrt{N}\bigr)$ (converted to k-CNF)

Conversion to k-CNF "Graph Versions" of Formulas

- Need bounded-width CNFs to use lower bound in [BW99]
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- **Solution:** Restrict formulas to bounded-degree graphs

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- Now width lower bounds $\Rightarrow$ size lower bounds
- And size lower bounds hold for original, unrestricted formulas

Polynomial Calculus (PC)

From [Clegg et al. '96] with adjustment in [Alekhnovich et al. '02]

Clauses interpreted as polynomial equations over field

Example: $x \vee y \vee \overline{z}$ gets translated to $xy\overline{z} = 0$
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Derivation rules

Boolean axioms $\dfrac{}{x^2 - x = 0}$

Negation $\dfrac{}{x + \overline{x} = 1}$

Linear combination $\dfrac{p = 0 \qquad q = 0}{\alpha p + \beta q = 0}$

Multiplication $\dfrac{p = 0}{xp = 0}$

Goal: Derive $1 = 0 \Leftrightarrow$ no common root $\Leftrightarrow$ formula unsatisfiable

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Clauses turn into monomials

Write out all polynomials as sums of monomials

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Size — analogue of resolution length/size
total # monomials in refutation counted with repetitions

Degree — analogue of resolution width
largest degree of monomial in refutation

Polynomial Calculus Strictly Stronger than Resolution

Polynomial calculus simulates resolution efficiently

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- Essentially no increase in length/size or width/degree
- Hence worst-case upper bounds for resolution carry over

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Polynomial calculus strictly stronger w.r.t. size and degree

- Tseitin formulas (over $\mathrm{GF}(2)$ can do Gaussian elimination)
- Onto functional pigeonhole principle (over any field) [Riis '93]
- Also other examples

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- Degree upper bound ⇒ size upper bound [Clegg et al.'96]
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- **Open problem:** Are functional PHP and onto PHP formulas hard for polynomial calculus?

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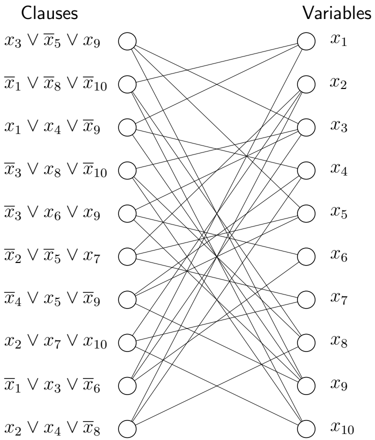

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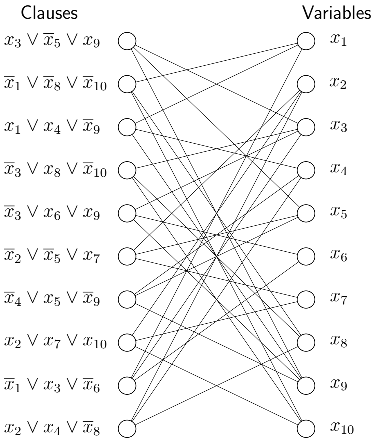

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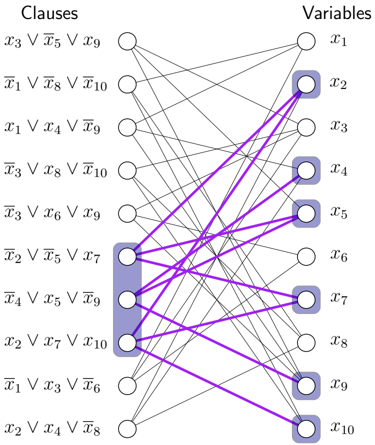

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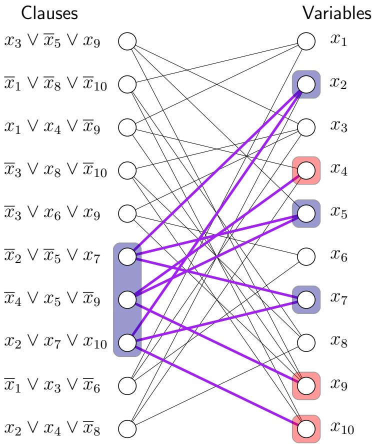

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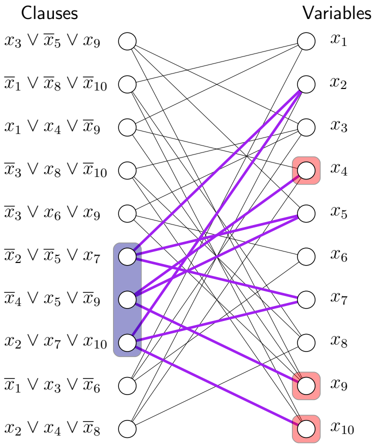

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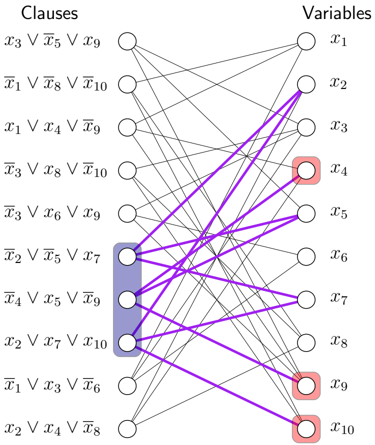

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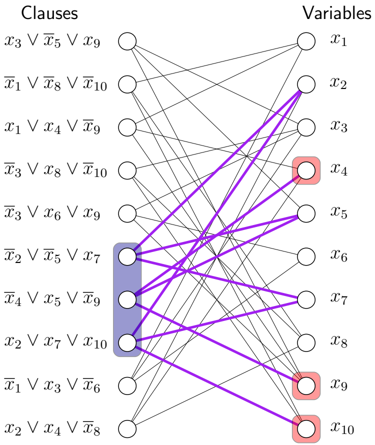
Need graph capturing combinatorial structure!



Generalized Incidence Graphs for CNF Formulas

Given CNF formula $\mathcal{F}$ over variables $\mathcal{V}$

- Partition clauses into $\mathcal{F} = \bigcup_{i=1}^{m} F_i \cup E$ (for E satisifiable)
- Divide variables into $\mathcal{V} = \bigcup_{j=1}^{n} V_j$ — **not** always partition
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Build bipartite $(\mathcal{U}, \mathcal{V})_E$-graph $\mathcal{G}$

- Left vertices $\mathcal{U} = \{F_1, \ldots, F_m\}$
- Right vertices $\mathcal{V} = \{V_1, \ldots, V_n\}$
- Edge (F_i, V_j) if $\mathit{Vars}(F_i) \cap V_j \neq \emptyset$
- Two types of edges depending on how F_i and V_j behave (modulo assignments α satisfying "filtering set" E)

The Importance of Basic Courtesy

$F \in \mathcal{U}$ and $V \in \mathcal{V}$ are E-semirespectful neighbours if

- given any total assignment α such that $\alpha(E) = 1$
- can flip α on V to α' so that $\alpha'(F \wedge E) = 1$

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Example

$F_1 = \{x \vee y,\ x \vee \overline{z},\ \overline{x} \vee z\}$, $V = \{x, y\}$, $E = \{\overline{y} \vee z\}$

Not E-semirespectful — consider $\alpha = \{y \mapsto 0, z \mapsto 0\}$

Not allowed to flip $z \notin V$; flipping y falsifies E; but $F_1\restriction_\alpha = \{x, \overline{x}\}$

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Example

Change to $F_2 = \{x \vee \overline{y},\, x \vee \overline{z},\, \overline{x} \vee y \vee z\}$, $V = \{x, y\}$, $E = \{\overline{y} \vee z\}$

Now F_2 and V E-semirespectful — given any α s.t. $\alpha(\overline{y} \vee z) = 1$

can always flip value assigned to x to $\alpha(y \vee z)$

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Example

Change to $F_2 = \{x \vee \overline{y},\ x \vee \overline{z},\ \overline{x} \vee y \vee z\}$, $V = \{x, y\}$, $E = \{\overline{y} \vee z\}$
Now F_2 and V E-semirespectful — given any α s.t. $\alpha(\overline{y} \vee z) = 1$
can always flip value assigned to x to $\alpha(y \vee z)$

(To simplify, think of all edges (F_i, V_j) as being E-semirespectful)

Semirespectful Expanders and Width Lower Bounds

Recall boundary $\partial(\mathcal{U}') = \{V \in \mathcal{N}(\mathcal{U}') \mid \mathcal{N}(V) \cap \mathcal{U}' = \{F\} \text{ unique}\}$

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An $(\mathcal{U}, \mathcal{V})_E$-graph is an (s, δ, E)-semirespectful expander if for all $\mathcal{U}' \subseteq \mathcal{U}$, $|\mathcal{U}'| \leq s$ it holds that $|\partial^{\mathsf{sr}}_E(\mathcal{U}')| \geq \delta|\mathcal{U}'|$

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Theorem (essentially [BW99])

If $\mathcal{F}$ has (s, δ, E)-semirespectful expander $(\mathcal{U}, \mathcal{V})_E$ with overlap ℓ, then

$$\textit{resolution proof width} > \frac{\delta s}{2\ell}$$

Progress Measure Approach (1/4)

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Proof: Define "progress measure" $\mu : \{\text{clauses}\} \rightarrow \mathbb{N}$ such that

1. $\mu(\text{axiom clause}) = \mathcal{O}(1)$
2. $\mu(C \vee D) \leq \mu(C \vee x) + \mu(D \vee \overline{x})$
3. $\mu(\bot) > s$

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$\Rightarrow$ in any resolution proof $\exists\, C$ with $\mu(C) = \sigma$ for $s/2 < \sigma \leq s$

$\Rightarrow$ such C has width $\geq \delta\sigma/\ell$ □

Progress Measure Approach (2/4)

Given (s, δ, E)-semirespectful expander $(\mathcal{U}, \mathcal{V})_E$ for $\mathcal{F}$, define

$$\mu(C) := \min\{|\mathcal{U}'| \,;\, \bigwedge_{F \in \mathcal{U}'} F \wedge E \vDash C\}$$

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1. $\mu(A) = \mathcal{O}(1)$ for axioms $A \in \mathcal{F} = \bigcup_{i=1}^{m} F_i \cup E$
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   - Fix minimal $\mathcal{U}_2$ s.t. $\bigwedge_{F \in \mathcal{U}_2} F \wedge E \vDash D \vee \overline{x}$
   - Then it holds that

$$\bigwedge_{F \in \mathcal{U}_1 \cup \mathcal{U}_2} F \wedge E \vDash C \vee D \ ,$$

so $\mu(C \vee D) \leq |\mathcal{U}_1 \cup \mathcal{U}_2| \leq |\mathcal{U}_1| + |\mathcal{U}_2| = \mu(C \vee x) + \mu(D \vee \overline{x})$

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3. $\mu(\bot) > s$ for empty clause $\bot$

Progress Measure Approach (3/4)

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   - By "peeling argument" $\exists$ matching $F_1 \leftrightarrow V_1, \ldots, F_s \leftrightarrow V_s$ s.t. $V_i \in \mathcal{N}(F_i) \setminus \mathcal{N}\left(\bigcup_{j=1}^{i-1} F_j\right)$ and F_i & V_i E-semirespectful

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   - So $\bigwedge_{F_i \in \mathcal{U}'} F_i \wedge E \nvDash \bot$ for $|\mathcal{U}'| \leq s$ and hence $\mu(\bot) > s$

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Given (s, δ, E)-semirespectful expander $(\mathcal{U}, \mathcal{V})_E$ with overlap ℓ

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Proof of claim: Another flipping argument using semirespectfulness

- Fix $V \in \partial^{\mathsf{sr}}_E(\mathcal{U}_C)$ and unique neighbour $F_V \in \mathcal{U}_C$ of V
- By minimality, $\exists\, \alpha$ s.t. $\alpha\big(\bigwedge_{F \in \mathcal{U}_C \setminus \{F_V\}} F \wedge E\big) = 1$ but $\alpha(C) = 0$
- If $V \cap \mathit{Vars}(C) = \emptyset$, then E-semirespectfully flip α on V to satisfy F_V ↯

Applications: Tseitin and Onto-FPHP

Tseitin formulas

- F_i = clauses encoding parity constraint for ith vertex
- V_j = singleton set with jth edge (so overlap $\ell = 1$)
- $E = \emptyset$
- If underlying graph edge expander, then $(\mathcal{U}, \mathcal{V})_E$-graph semirespectful boundary expander with same parameters

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Onto functional PHP formulas

- F_i = singleton set with pigeon axiom for pigeon i
- V_j = all variables $p_{i,j}$ mentioning hole j (again overlap $\ell = 1$)
- E = all hole, functional, and onto axioms
- If onto FPHP restricted to bipartite graph, then $(\mathcal{U}, \mathcal{V})_E$-graph semirespectful boundary expander with same parameters

From Resolution to Polynomial Calculus

Obtain resolution width lower bounds from expander graphs where we can win following game on edges

Resolution edge game on (F, V) with side constraints E

1. Adversary provides total assignment α such that $\alpha(E) = 1$
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But Tseitin and onto FPHP both easy for polynomial calculus!

So semirespectful boundary expanders cannot yield any lower bounds for polynomial calculus

A Harder Edge Game for Polynomial Calculus

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To get polynomial calculus degree lower bounds need winning strategy for harder game on expander graphs

Polynomial calculus edge game on (F, V) with side constraints E

1. Commit to $\alpha_V : V \to \{0, 1\}$
2. Adversary provides total assignment α such that $\alpha(E) = 1$
3. Flipping α on V to α_V should yield $\alpha[\alpha_V/V](F \wedge E) = 1$

Fully Respectful Neighbours

$F \in \mathcal{U}$ and $V \in \mathcal{V}$ are E-respectful neighbours if possible to find $\alpha_V : V \to \{0,1\}$ such that

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Example

$F_2 = \{x \vee \overline{y},\ x \vee \overline{z},\ \overline{x} \vee y \vee z\}$, $V = \{x, y\}$, $E = \{\overline{y} \vee z\}$
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(Also holds for sets of polynomials not obtained from CNFs)

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- CNFs with expanding CVIGs [Alekhnovich & Razborov '01]
- "Vanilla" PHP formulas [Alekhnovich & Razborov '01]
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New contribution: Functional PHP is hard

Hardness of Different Flavours of PHP

Variant	Resolution	Polynomial calculus
PHP		
FPHP		
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Onto-PHP	hard [Hak85]	hard [AR01]
Onto-FPHP	hard [Hak85]	easy! [Rii93]

This work

- Observe that [AR01] proves hardness of Onto-PHP

Hardness of Different Flavours of PHP

Variant	Resolution	Polynomial calculus
PHP	hard [Hak85]	hard [AR01]
FPHP	hard [Hak85]	hard [MN15]
Onto-PHP	hard [Hak85]	hard [AR01]
Onto-FPHP	hard [Hak85]	easy! [Rii93]

This work

- Observe that [AR01] proves hardness of Onto-PHP
- Prove that FPHP is hard in polynomial calculus

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Theorem ([MN15])

If G is a (standard) bipartite (s, δ)-boundary expander with left degree $\leq d$, then $FPHP_G$ requires PC degree $> \delta s/(2d)$.

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- So get same expansion parameters, and theorem follows ☐

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- Go beyond polynomial calculus (e.g. to Positivstellensatz)

Take-away Message

Generalized method for PC degree lower bounds

- Unified framework for most previous lower bounds
- Exponential size lower bound for functional PHP

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- Develop non-degree-based size lower bound techniques

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Thank you for your attention!