Lower Bounds for Monotone Models (by Algebraic Gaps)

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Familiar Picture

$$\mathsf{NC}^1 \subseteq \mathsf{L} \subseteq \mathsf{NL} \subseteq \mathsf{NC} \subseteq \mathsf{P}$$

Familiar Picture

Consider the corresponding (polynomial-size) circuit models capturing the classes.

$$NC^1 \subseteq L \subseteq NL \subseteq NC \subseteq P$$

Familiar Picture

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Formulas

Familiar Picture

$$\mathrm{NC}^1 \subseteq \mathrm{L} \subseteq \mathrm{NL} \subseteq \mathrm{NC} \subseteq \mathrm{P}$$

Formulas

$\mathrm{NC}^1(f) =$ formula size of f

Familiar Picture

Switching Networks
(Branching Programs)

$$NC^1 \subseteq L \subseteq NL \subseteq NC \subseteq P$$

Formulas

Familiar Picture

Switching Networks
(Branching Programs)

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Formulas

Directed Switching Networks
(Non-det. Branching Programs)

Familiar Picture

Switching Networks
(Branching Programs)

Polylog-depth
Circuits

$$NC^1 \subseteq L \subseteq NL \subseteq NC \subseteq P$$

Formulas

Directed Switching Networks
(Non-det. Branching Programs)

Familiar Picture

Switching Networks (Branching Programs) → L

Polylog-depth Circuits → NC

$$NC^1 \subseteq L \subseteq NL \subseteq NC \subseteq P$$

Formulas → NC^1

Directed Switching Networks (Non-det. Branching Programs) → NL

Circuits → P

Familiar Picture

$$NC^1 \subseteq L \subseteq NL \subseteq NC \subseteq P$$

Familiar Picture

How many separations do we have?

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Fortunately, this is easy to fix.

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$$\mathsf{mNC}^1 \subsetneq \mathsf{mL} \subsetneq \mathsf{mNL} \subsetneq \mathsf{mNC} \subsetneq \mathsf{mP}$$

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Monotone = No Negations in Circuit Models

Familiar Picture

How many separations do we have?

How did this picture come about?

$$\mathsf{mNC}^1 \subsetneq \mathsf{mL} \subsetneq \mathsf{mNL} \subsetneq \mathsf{mNC} \subsetneq \mathsf{mP}$$

Familiar Picture

$$\mathsf{mNC}^1 \subsetneq \mathsf{mL} \subsetneq \mathsf{mNL} \subsetneq \mathsf{mNC} \subsetneq \mathsf{mP}$$

Karchmer-Wigderson '88
(Undirected st-connectivity)

Familiar Picture

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(Undirected st-connectivity)

Raz-Mckenzie '97
(GEN)

Familiar Picture

Potechin '10
(Directed st-connectivity)

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Raz-Mckenzie '97
(GEN)

Familiar Picture

There is a complexity measure which can lower bound all of these models!

$$\mathsf{mNC}^1 \subsetneq \mathsf{mL} \subsetneq \mathsf{mNL} \subsetneq \mathsf{mNC} \subsetneq \mathsf{mP}$$

Rank Measure

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$$f : \{0,1\}^n \to \{0,1\}$$

Rank Measure

$f : \{0,1\}^n \to \{0,1\}$

f monotone

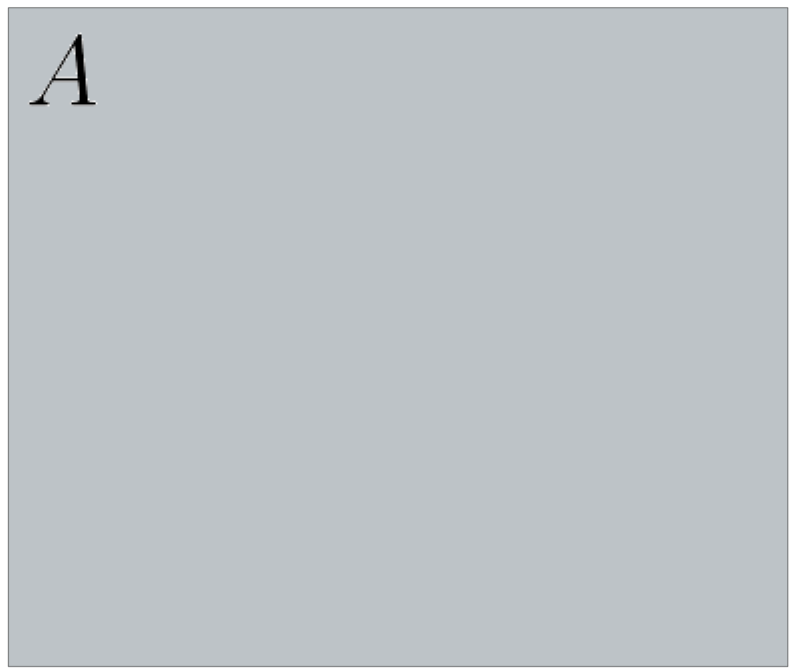
Rank Measure

$f : \{0,1\}^n \rightarrow \{0,1\}$

f monotone

$f^{-1}(0)$

$f^{-1}(1)$



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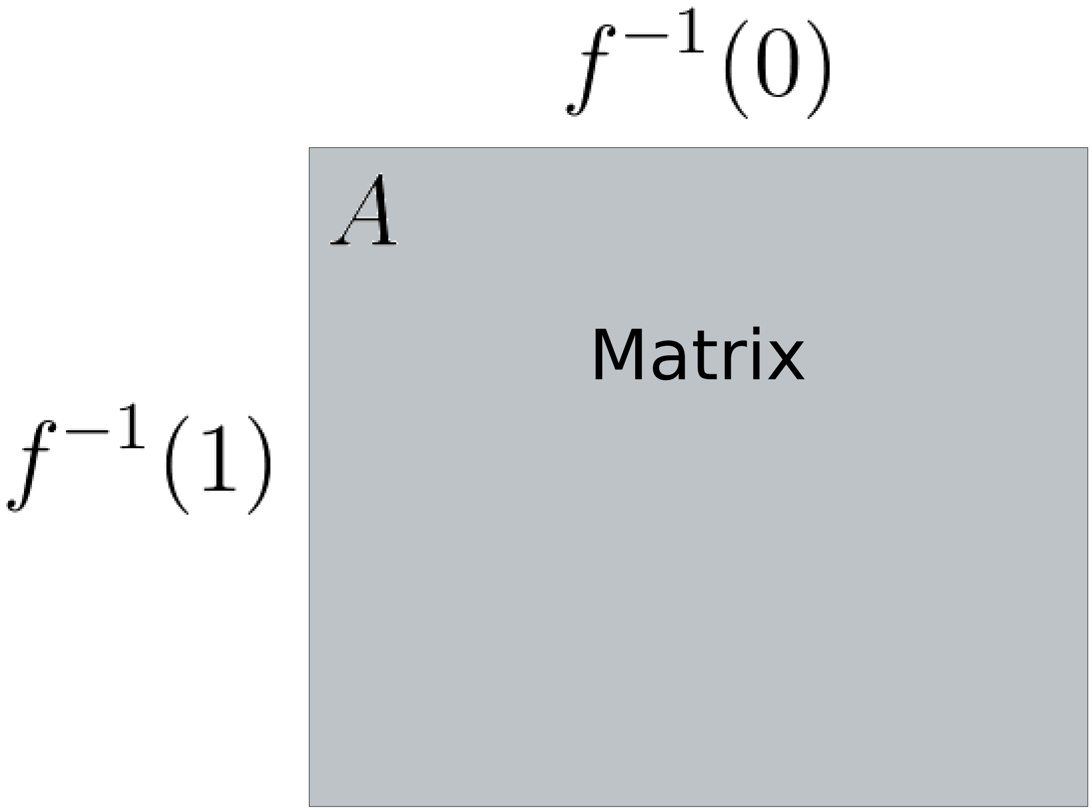

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A

Matrix

Not Communication Matrix

Rank Measure

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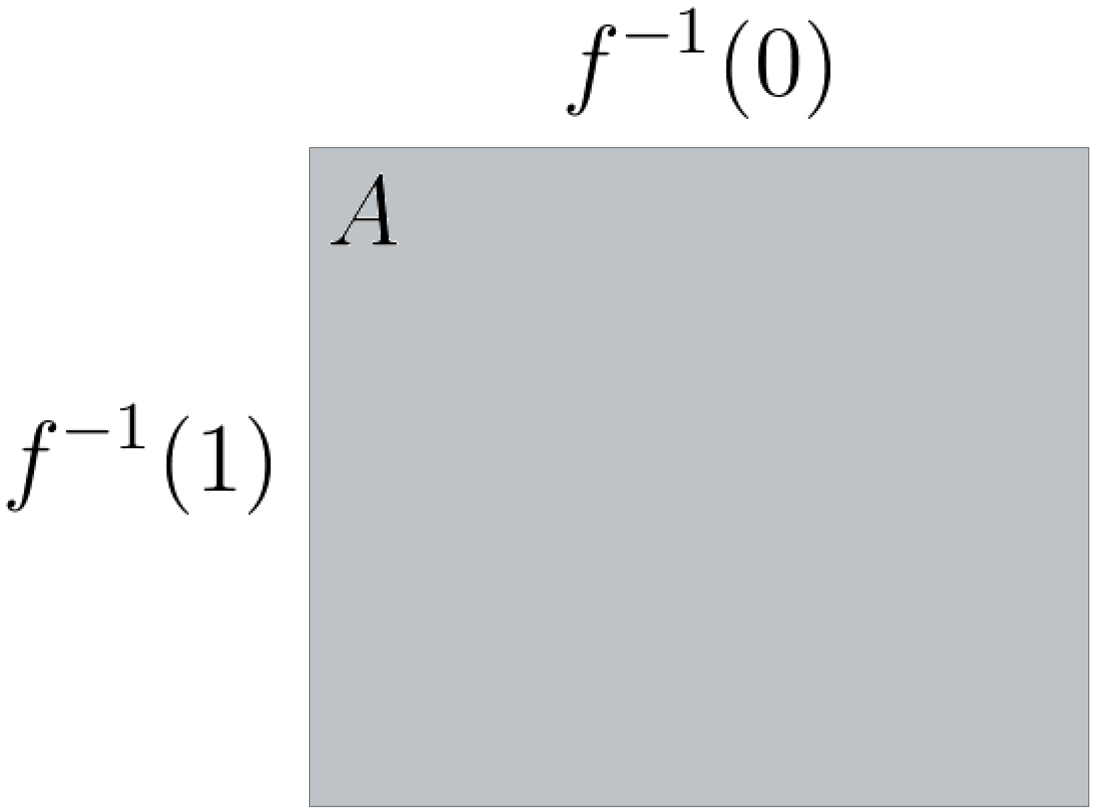

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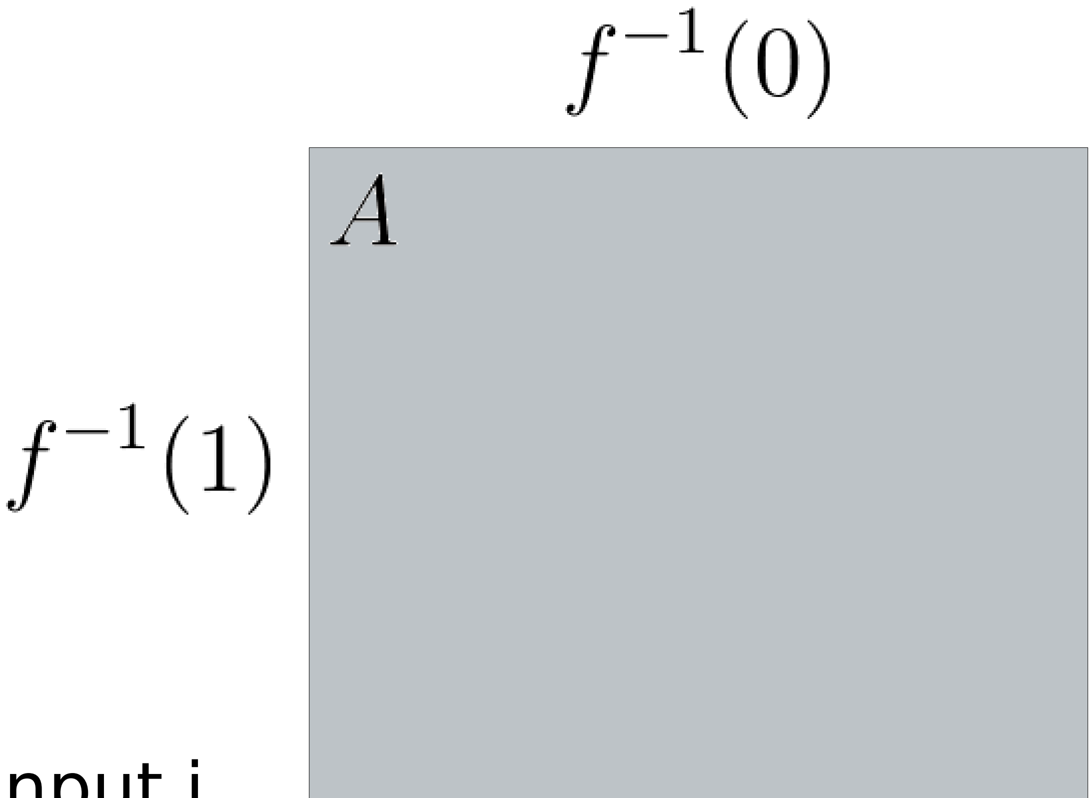

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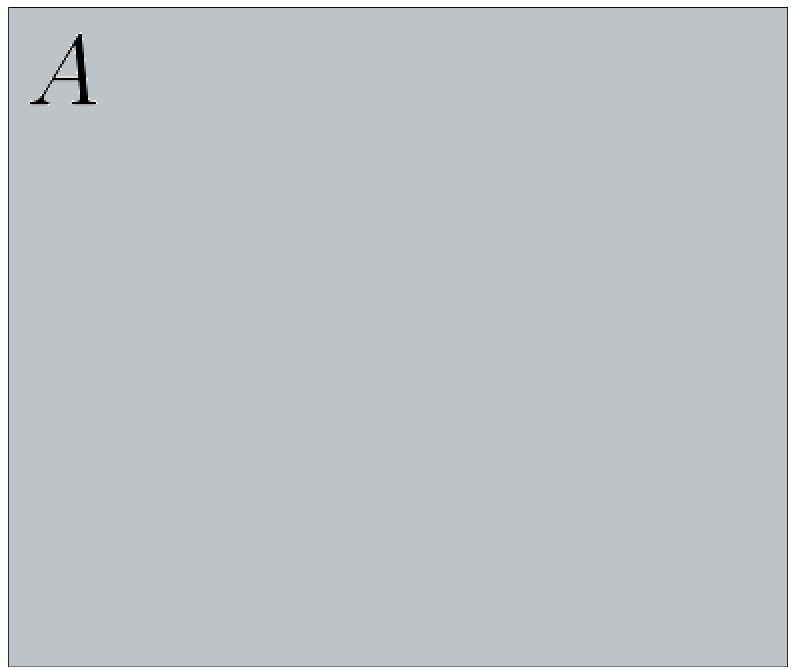

For any input i,

$$R_i = \{(x,y) \in f^{-1}(1) \times f^{-1}(0) \mid x_i = 1, y_i = 0\}$$

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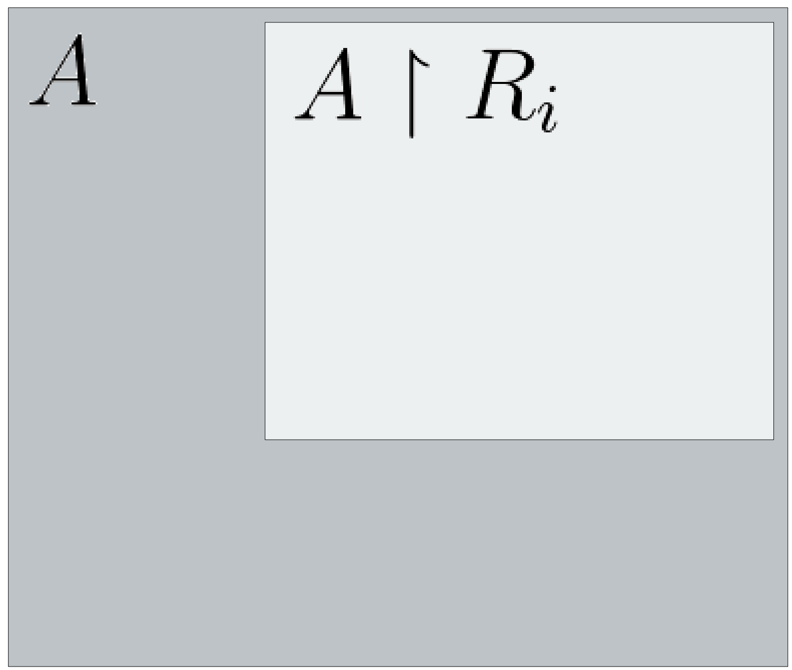

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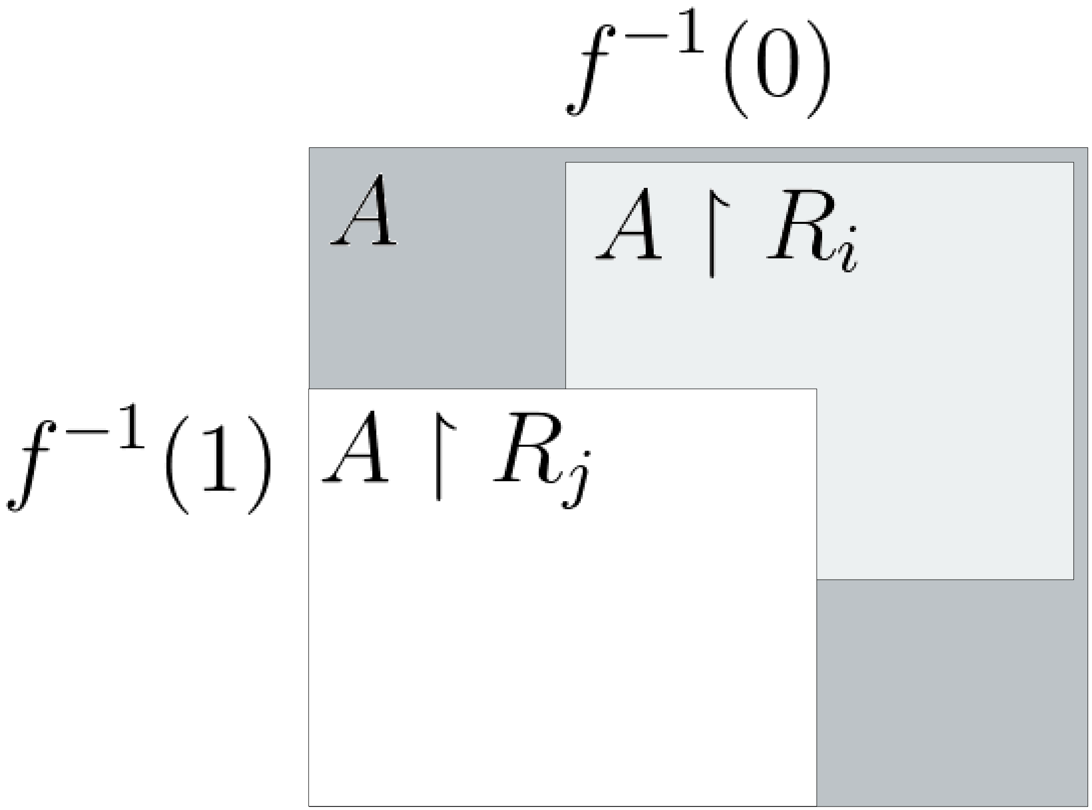

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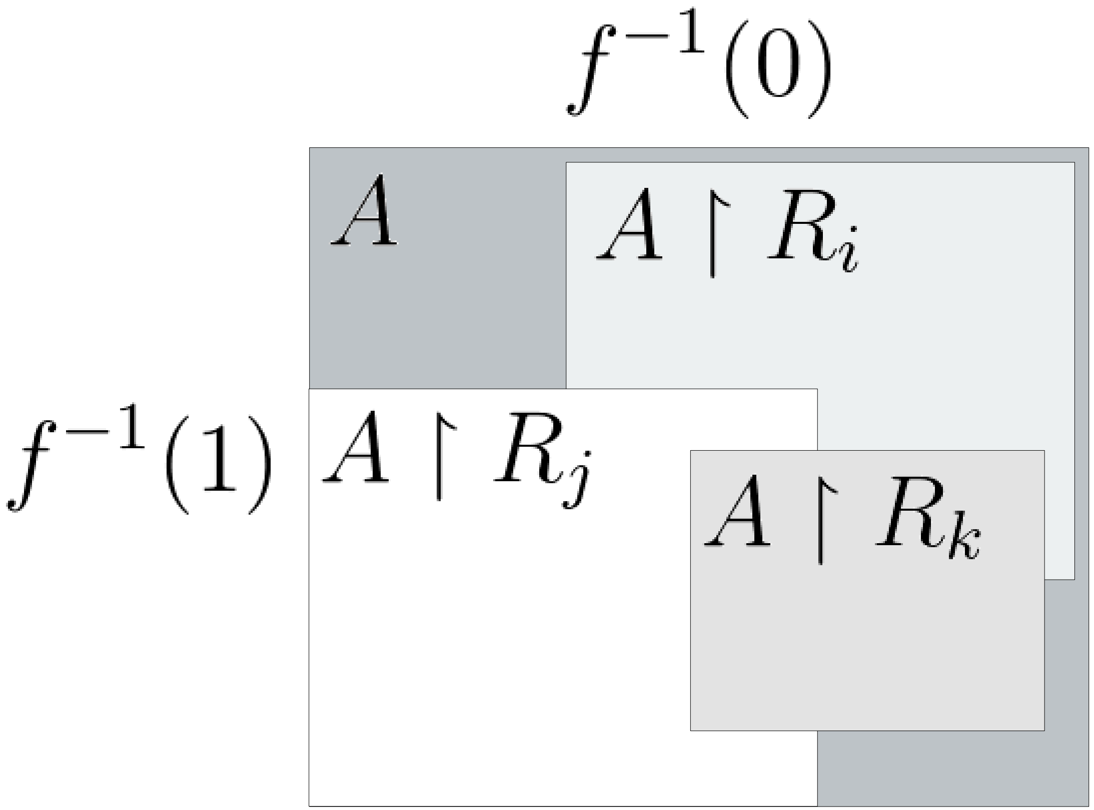

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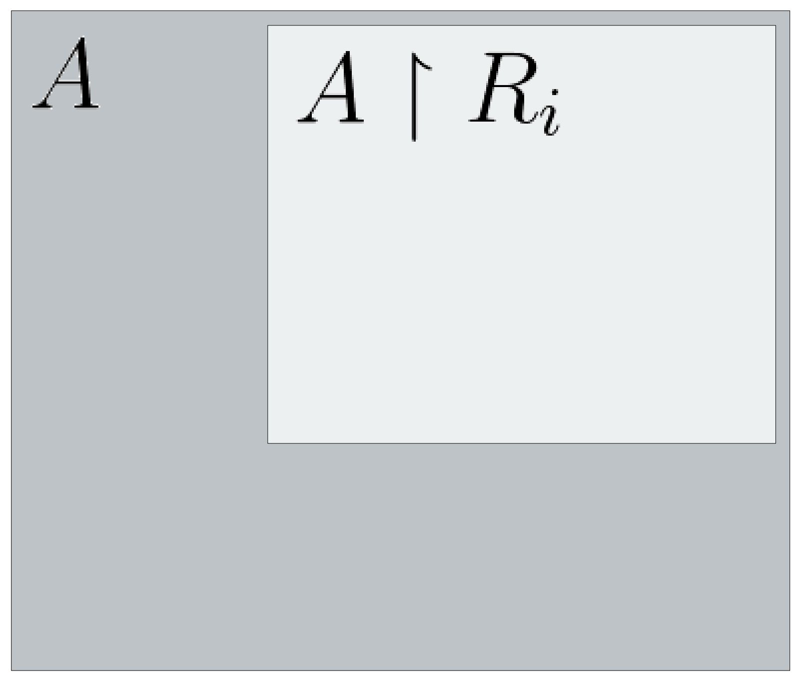

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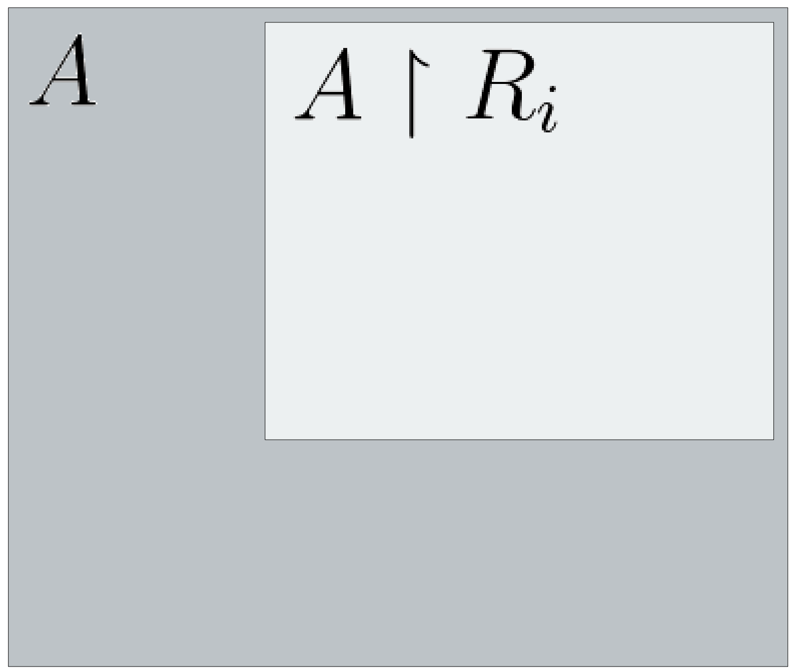

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Rank Measure [Razborov '90]:

$$\mu_A(f) = \frac{\text{rank}(A)}{\max_{i \in [n]} \text{rank}(A \restriction R_i)}$$

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Theorem [Razborov '90, KW '90, Gal '98]**:**
For any field **F**, any boolean function f,
and any matrix A over **F**,

$$\mu_A(f) \leq \mathrm{mSPAN}_{\mathbf{F}}(f) \leq \mathrm{mL}(f) \leq \mathrm{mNC}^1(f)$$

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Best prior lower bounds:

$N^{\Omega(\log N)}$ for a monotone function in NP.
[Razborov '90]

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$$\mathrm{mL} \subseteq \mathrm{mSPAN}_{\mathbf{F}}$$

-

-

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$$\cap|$$

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-

-

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$$\cap\!|$$

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[Babai et al '96] Quasipolynomial lower bounds against mNP.

-

-

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[BW '05] Quasipolynomial against nonmonotone NC

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Equivalent to Linear Secret Sharing Schemes. [KW '90]

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- First exponential lower bounds for monotone span programs.
- First separation between monotone span programs and mP/mNL/non-monotone span programs.
- ...

The Proof

The Proof

Lifting Theorem

(Communication Setting)

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(Communication Setting)

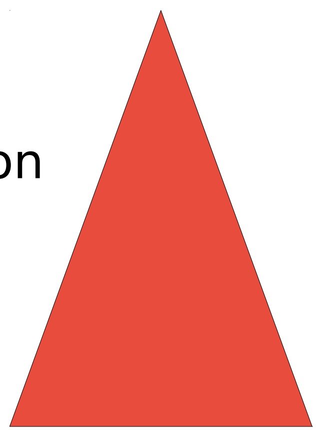
$f : \{0,1\}^n \to \{0,1\}$

Lifting Theorem

(Communication Setting)

$$f : \{0,1\}^n \to \{0,1\}$$

Decision Tree



Hard for
Weak Complexity
Measure

The Proof

Lifting Theorem

(Communication Setting)

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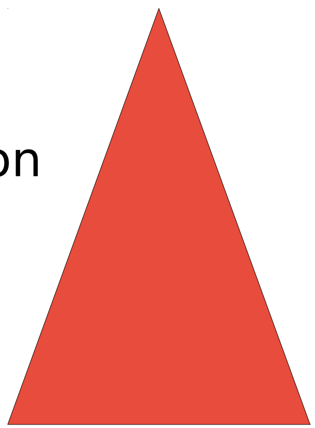
Decision Tree



$$f(g(x_1, y_1), g(x_2, y_2), \cdots, g(x_n, y_n))$$

Compose f with some "complex gadget" g

Hard for Weak Complexity Measure

The Proof

Lifting Theorem

(Communication Setting)

$f:\{0,1\}^n \rightarrow \{0,1\}$

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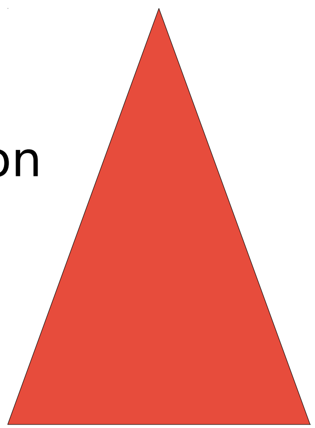


$f(g(x_1,y_1),g(x_2,y_2),\cdots,g(x_n,y_n))$

Compose f with some “complex gadget” g

Communication Complexity

Hard for Weak Complexity Measure

Hard for Strong Complexity Measure

The Proof

Lifting Theorem

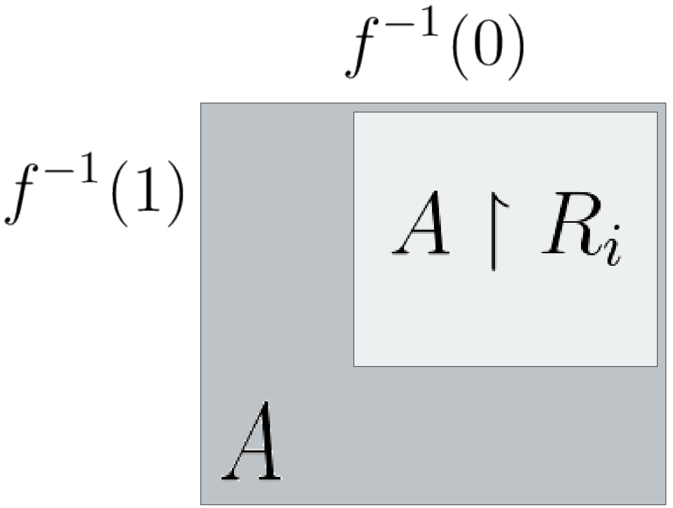
(Our Setting)

The Proof

Lifting Theorem

(Our Setting)

"Strong"



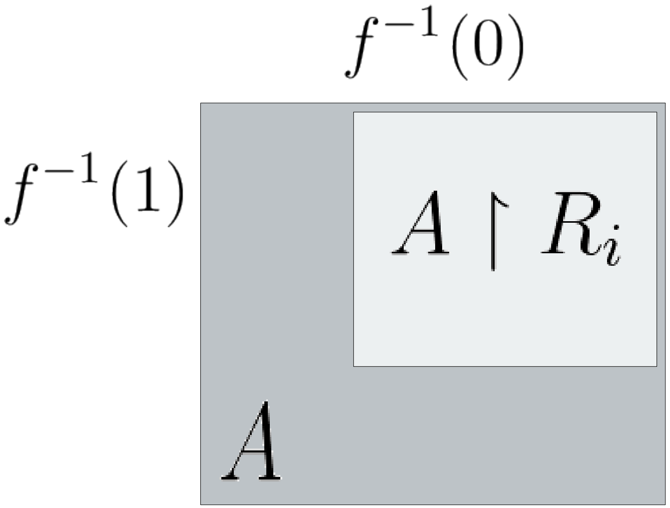
$$\mu_A(f) = \frac{\mathrm{rank}(A)}{\max_{i \in [n]} \mathrm{rank}(A \restriction R_i)}$$

The Proof

Lifting Theorem
(Our Setting)

"Weak?"

"Strong"



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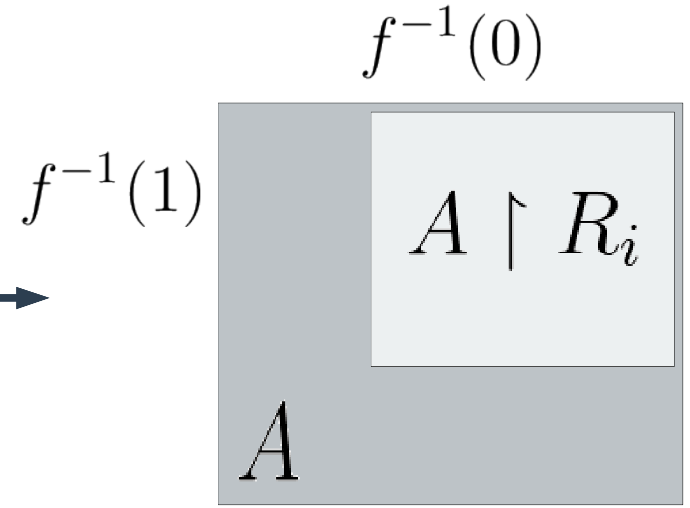

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The Proof

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(Our Setting)

"Weak?"

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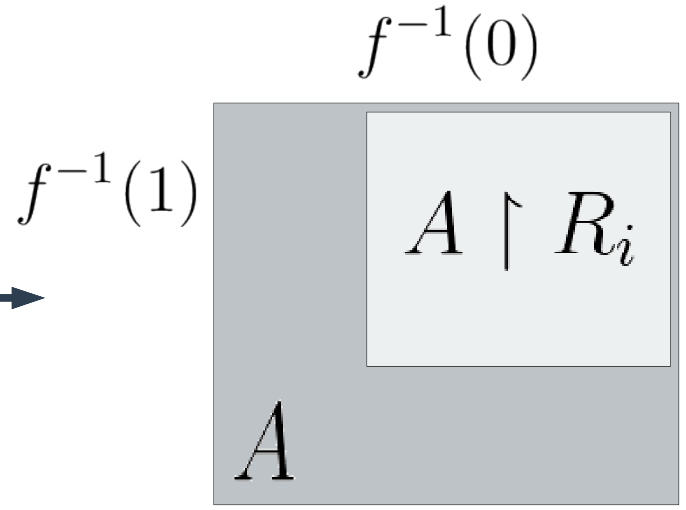
Algebraic Gaps

Complexity
Measure
On
Search Problems

?



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The Proof

Lifting Theorem
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“Weak?”

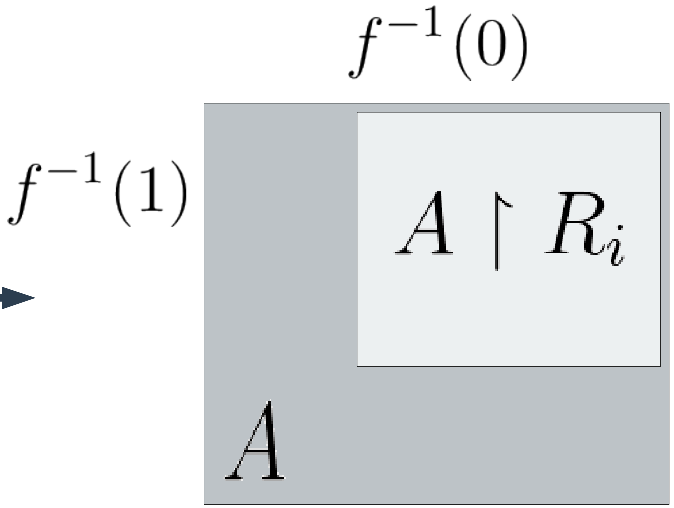
Algebraic Gaps

Complexity
Measure
On
Search Problems

“Pattern Matrices”
[Sherstov '08]

→

“Strong”



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Algebraic Gaps

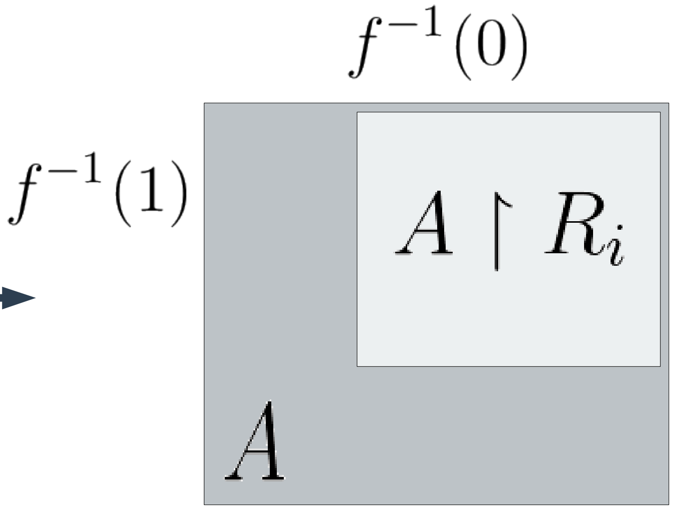
Complexity
Measure
On
Search Problems

"Pattern Matrices"
[Sherstov '08]

→

(Not just PMM,
need more
careful analysis)

"Strong"



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Algebraic Gaps (Definition)

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Ex. Fix an unsatisfiable CNF F. Given an assignment x, output the index of an unsatisfied clause of F.

<u>Def</u>: Let $S \subseteq Q \times \{-1, 1, *\}^n$ be a total search problem. The **algebraic gap complexity,** gap(S), of S is the maximum k for which there is a polynomial $p : \{-1, 1\}^n \to \mathbb{R}$ such that

$$\deg(p) = n, \quad \deg(p\restriction_C) \leq n - k$$

for each valid output C of S.

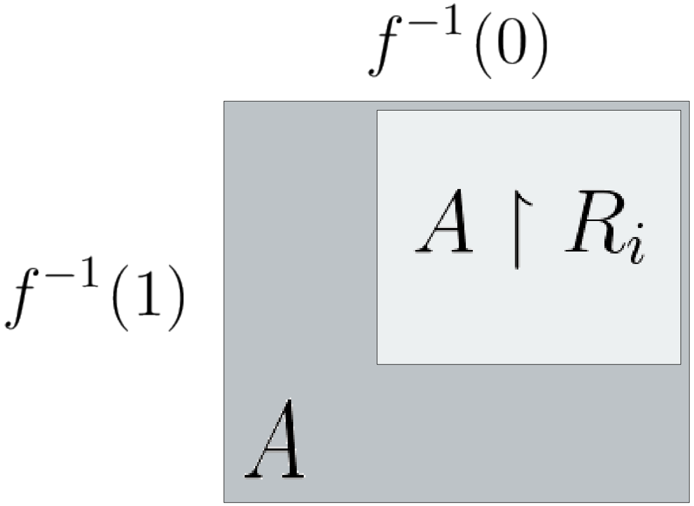
The Proof

“Weak?”

Lifting Theorem
(Our Setting)

“Strong”

Algebraic Gaps



The Proof

“Weak?”

Lifting Theorem

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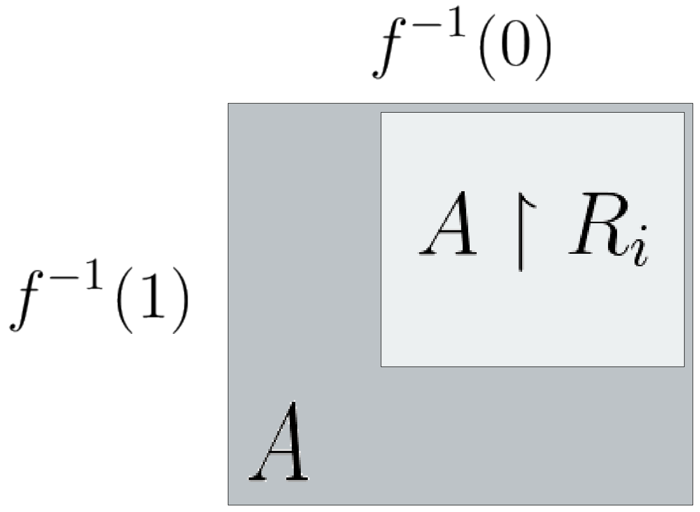
Algebraic Gaps →



Search Problem

The Proof

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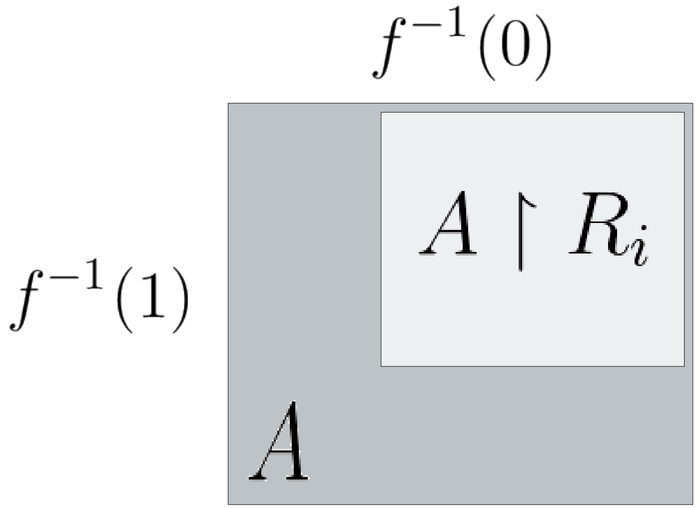


Search Problem → $f : \{0,1\}^n \to \{0,1\}$

The Proof

“Weak?”

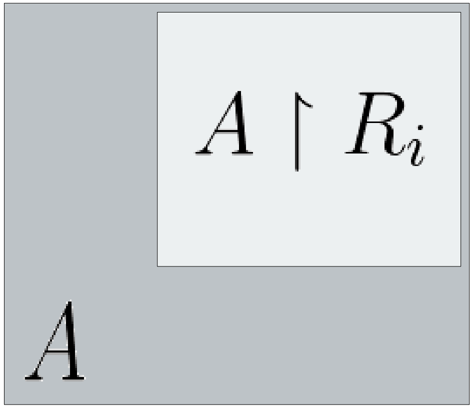
Lifting Theorem
(ST-CONNN)

“Strong”

Algebraic Gaps →

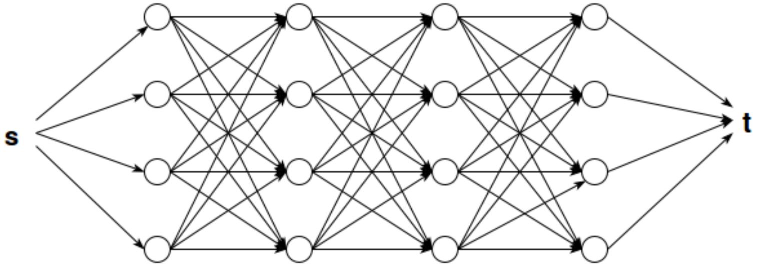
$f^{-1}(0)$

$f^{-1}(1)$



Search Problem →

STCONN



The Proof

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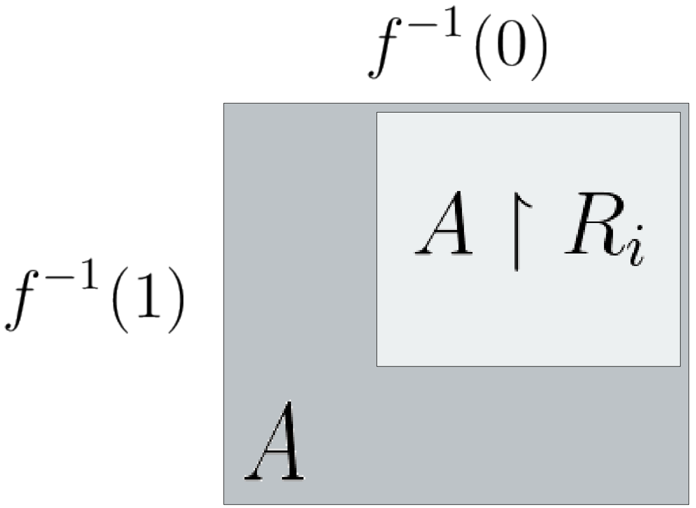

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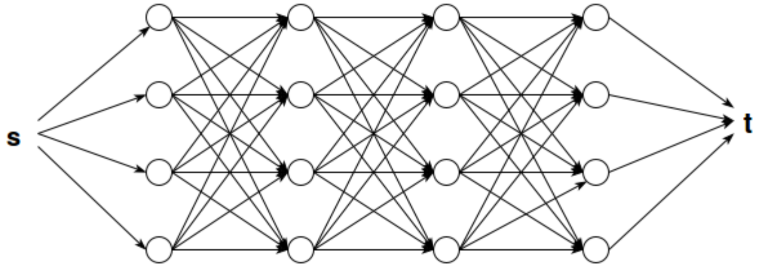


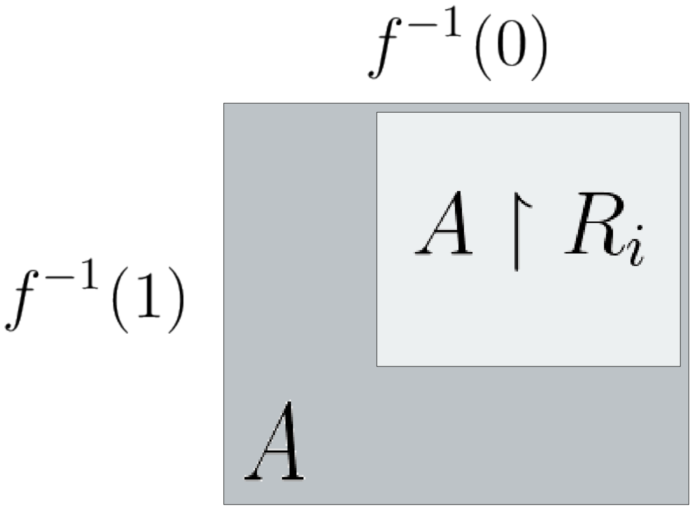
The Proof

“Weak?”

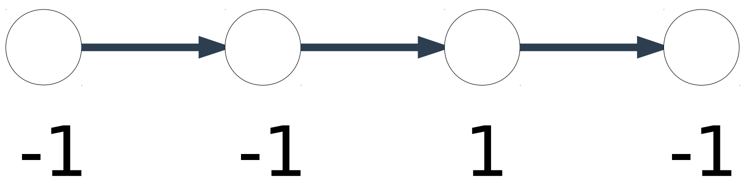
Lifting Theorem
(ST-CONN)

“Strong”

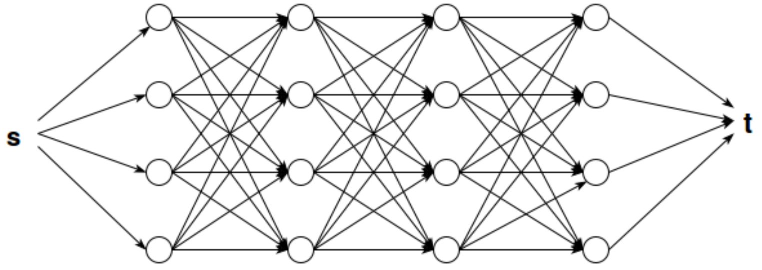
Algebraic Gaps



Search Problem



STCONN



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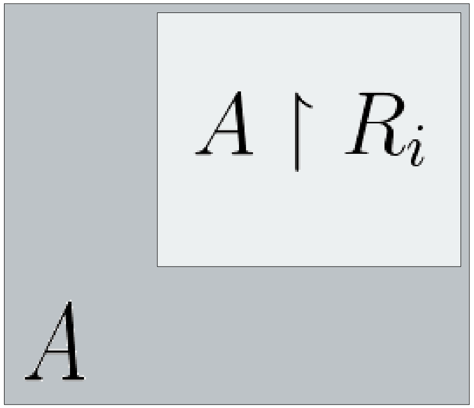

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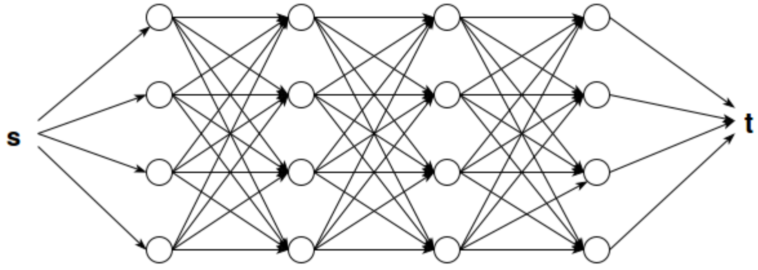

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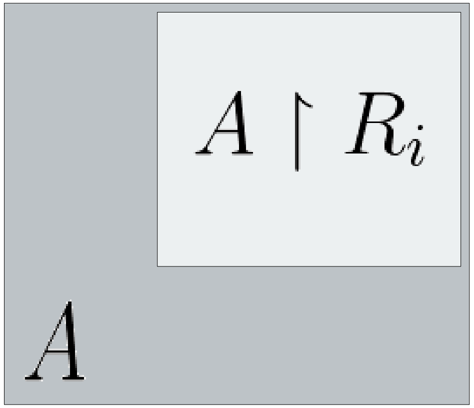

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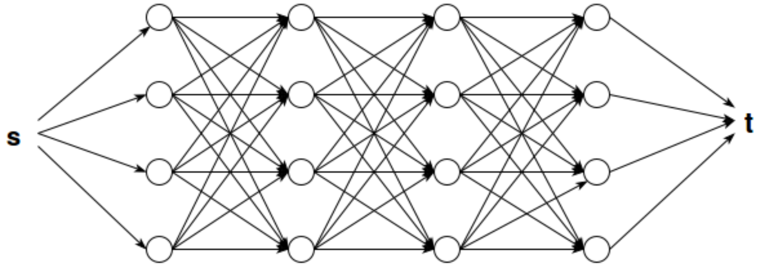

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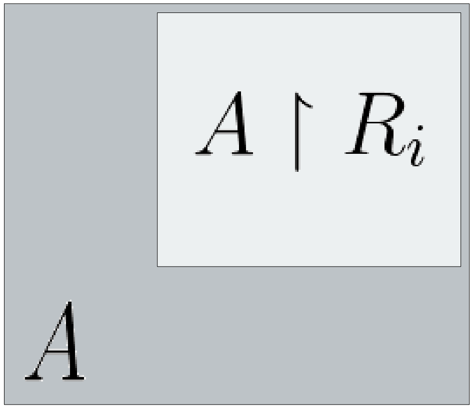

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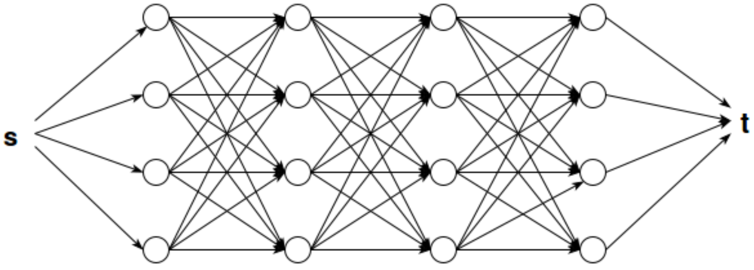

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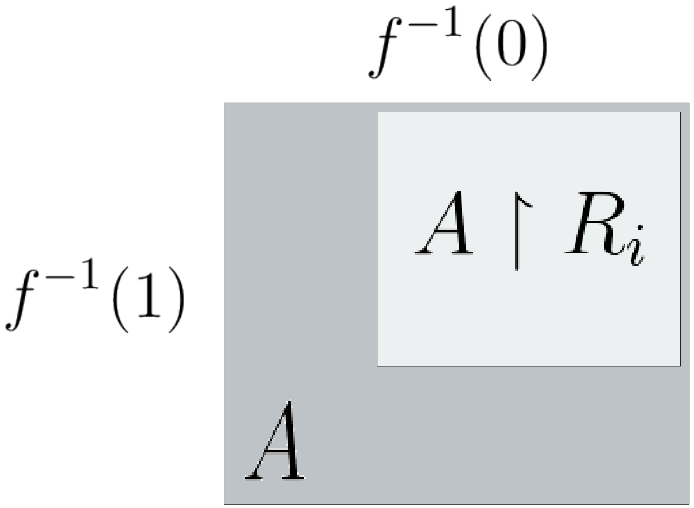

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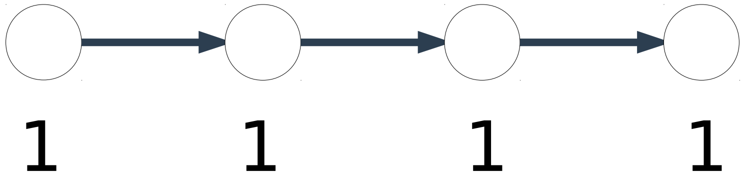

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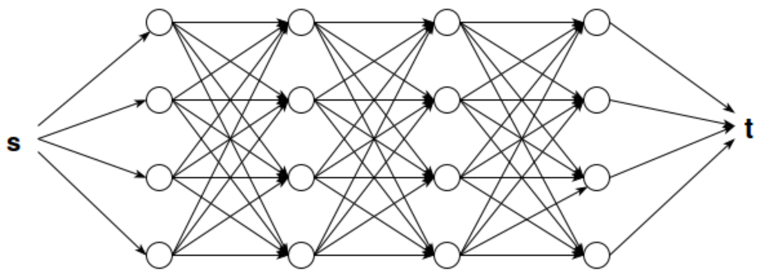

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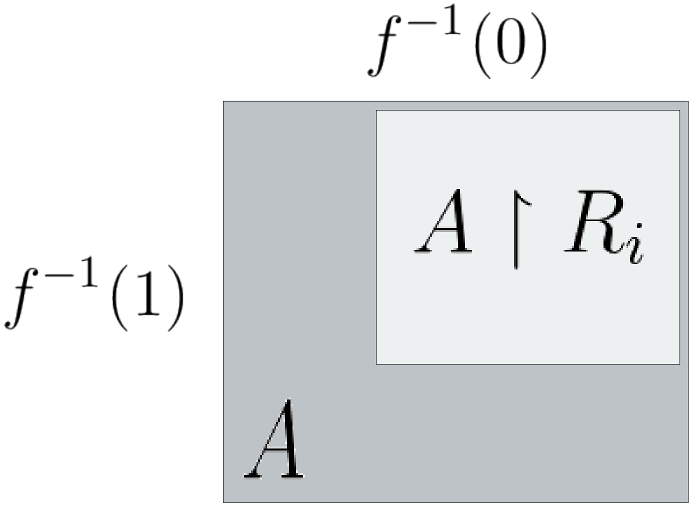

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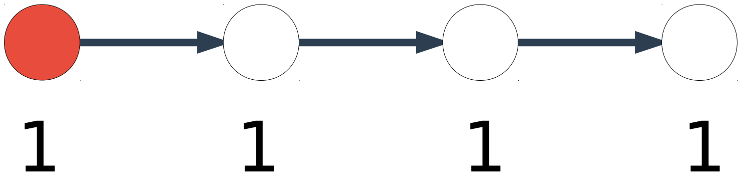

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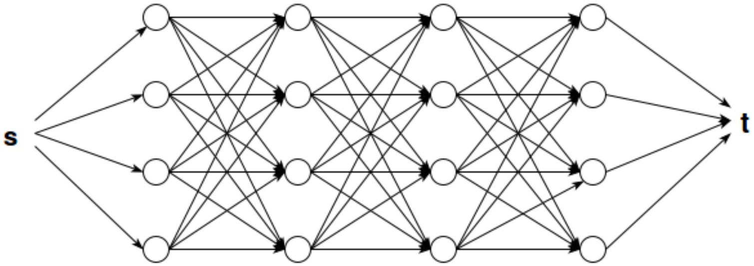

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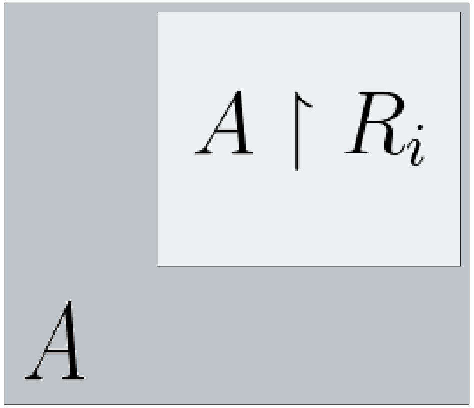

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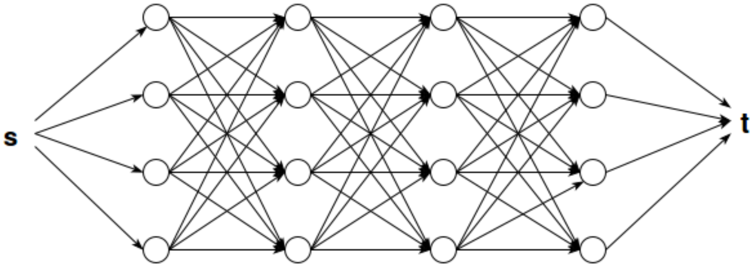

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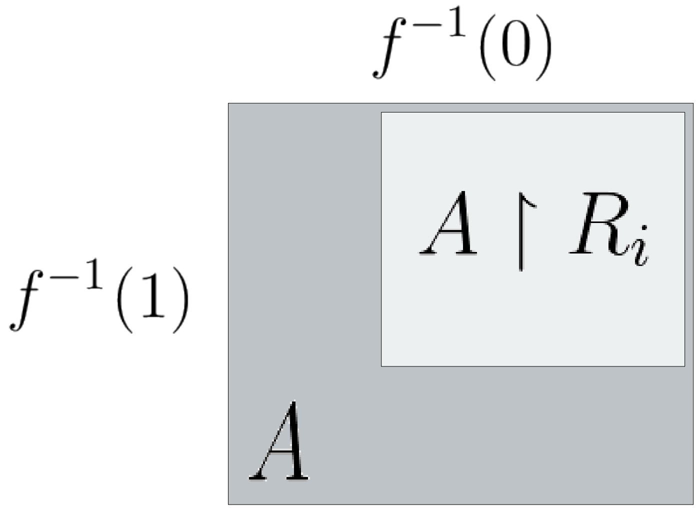
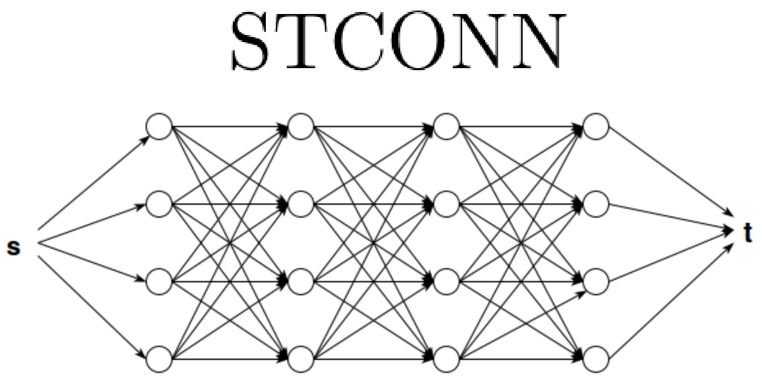
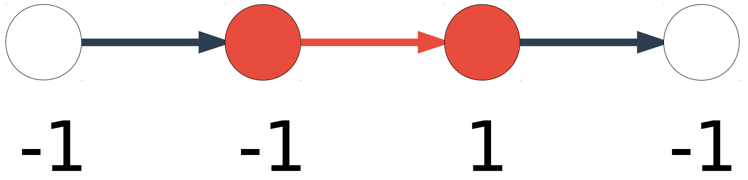


Search Problem

[Raz-Mckenzie '98]





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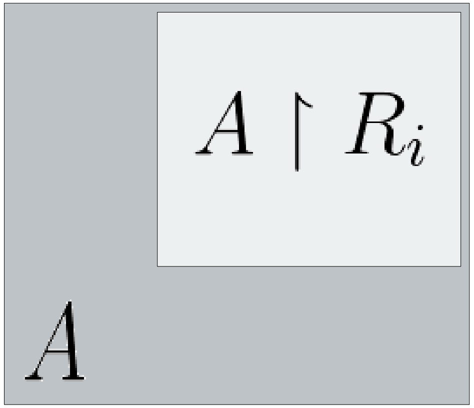

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$$p : \{-1, 1\}^n \to \mathbb{R}$$

$$\deg(p) = n \quad \deg(p\restriction_C) \leq n - k$$

[Raz-Mckenzie '98]

Search Problem

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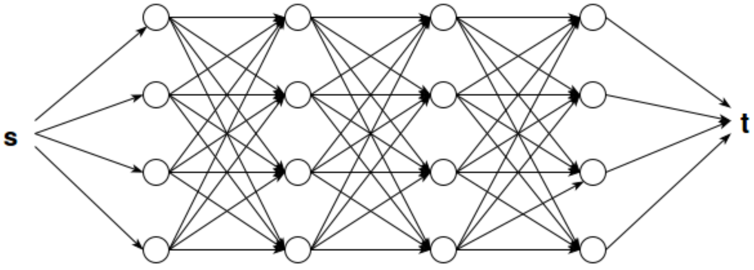

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$p(g(x_1, y_1), g(x_2, y_2), \ldots, g(x_n, y_n)))$

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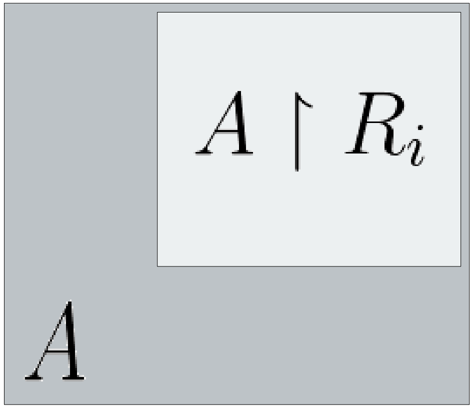

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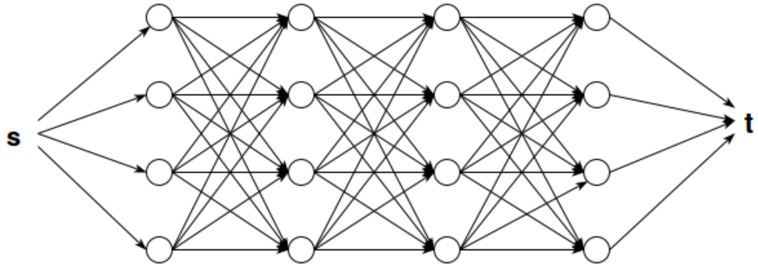

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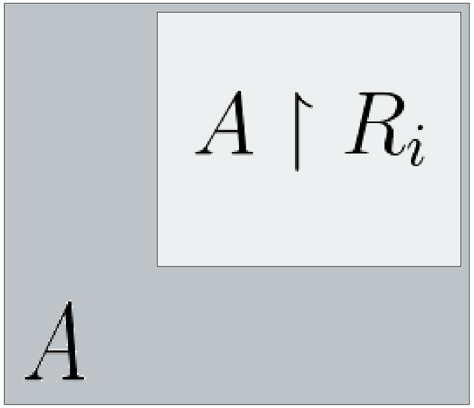

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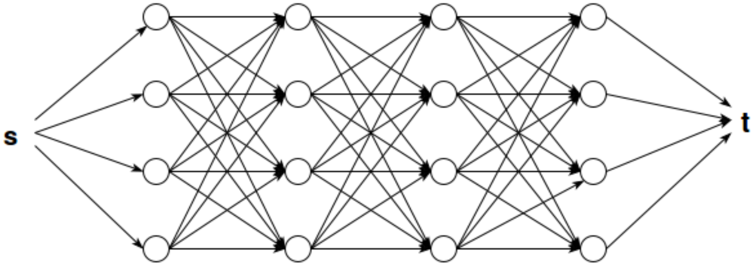

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Conclusion

Unified lower bounds against monotone models by "lifting".

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Algebraic gaps → other applications?

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Average case lower bounds?

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Algebraic gaps → other applications?

Average case lower bounds?

Sharpen lifting theorems further?

References

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