Average case lower bounds for threshold circuits

Ruiwen Chen, Rahul Santhanam and Srikanth Srinivasan

Oxford University and Department of Mathematics, IIT Bombay

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- Circuit computing function $f : \{0,1\}^n \to \{0,1\}$.
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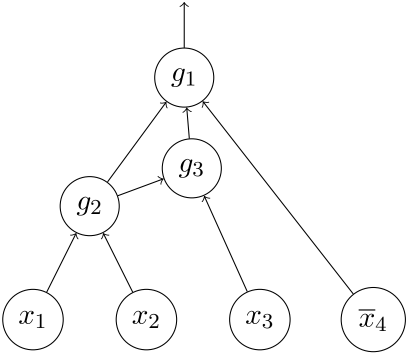

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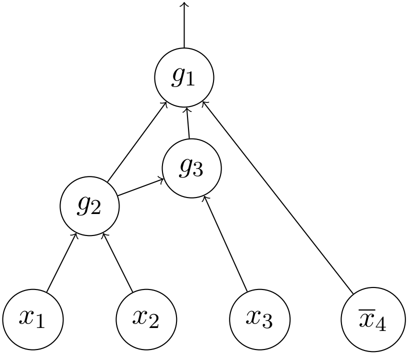

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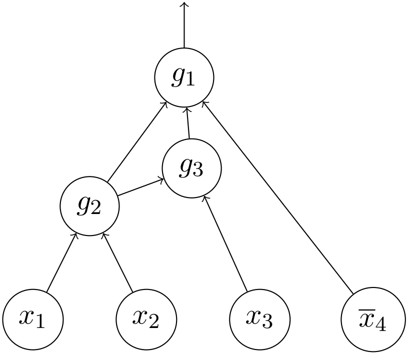
- Designated output gate computes function f.



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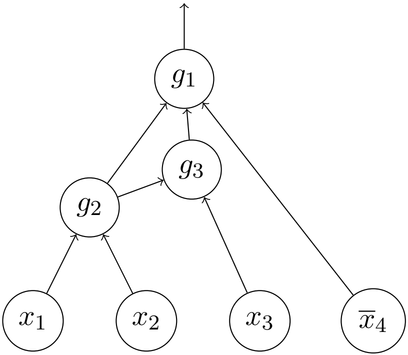

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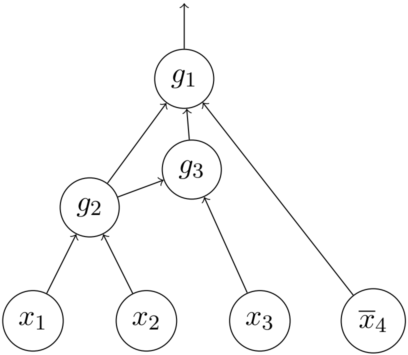

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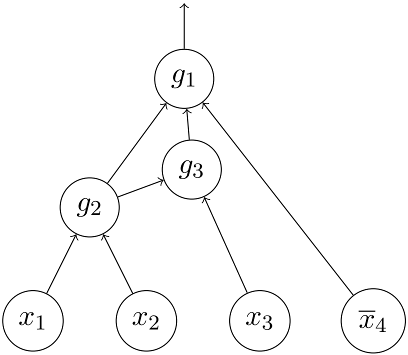

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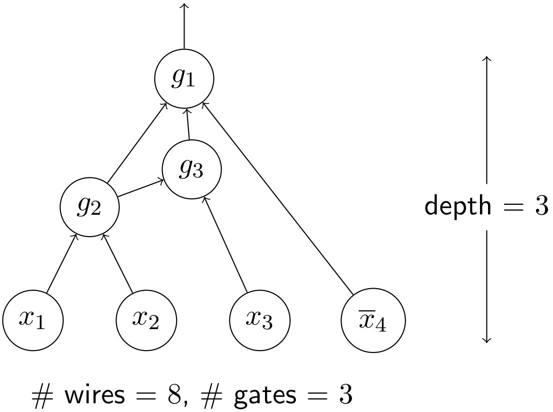

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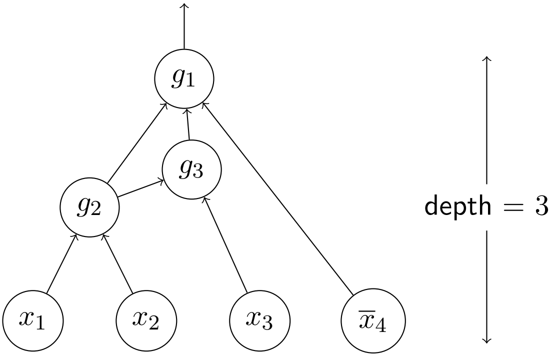

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- Generalize AC^0 circuits made up of AND and OR gates.

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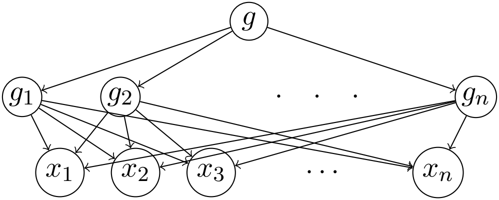

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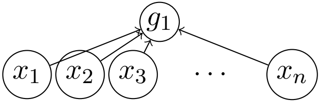

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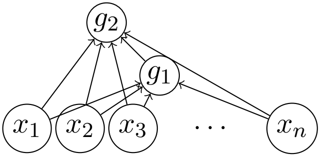

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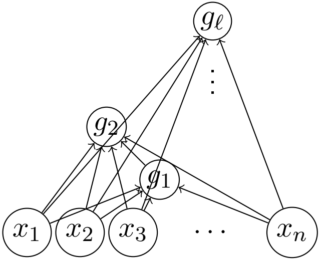

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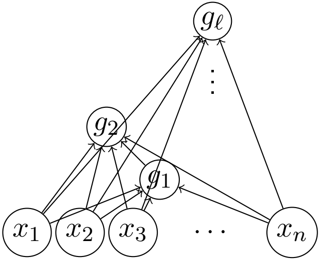

- General d: interpolate between the above two strategies.

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- Trick:

$$x_1 \oplus \cdots \oplus x_n = (x_1 \oplus \cdots \oplus x_m) \oplus (x_{m+1} \oplus \cdots \oplus x_{2m}) \oplus \cdots$$

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Work on threshold circuits

- Hajnal Maass Pudlák Turan Szegedy 1987
- (Polynomial Approximations) Paturi Saks 1991, Siu Roychowdhury Kailath 1992; Beigel 1994; Aspnes, Beigel, Furst and Rudich 1994; Podolskii 2012
- (Combinatorial restrictions) Impagliazzo Paturi Saks 1991
- (Communication complexity) Nisan 1992; Hansen and Miltersen 2004; Chattopadhyay and Hansen 2005; Lovett, S. 2012
- (Analytic techniques) Gopalan and Servedio 2010

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- Want to show that f hard on average against $\mathrm{TC}^0(s,d)$.

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- Prerequisite for constructing Pseudorandom generators (PRGs) for the circuit class.
- Increased understanding can lead to satisfiability algorithms, learning algorithms,...

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- Result 2: Different explicit function has exponentially small correlation with $\mathrm{TC}^0_w(n^{1+\delta^d}, d)$.

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- Random restriction $\rho \sim \mathcal{R}_p$:

$$\Pr_{\rho}[\rho(x_i) = *] = p \qquad \Pr_{\rho}[\rho(x_i) = 0/1] = \frac{1-p}{2}$$

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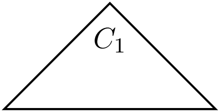

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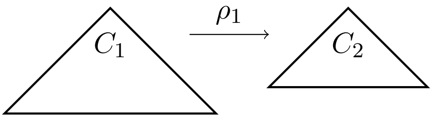

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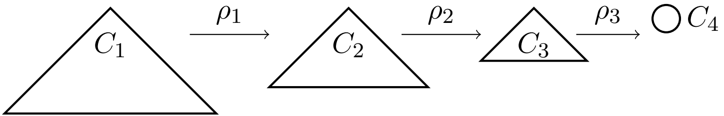
Random restrictions and lower bounds

- Role of Random restriction: simplify circuit, while leaving hard function (rel.) unchanged.
- Eg. with AC^0 and PARITY.



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- For threshold circuits: Peres' theorem.

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f a threshold function. $\mathrm{Corr}(f, \mathrm{PARITY}) \leq O(\frac{1}{\sqrt{n}})$.

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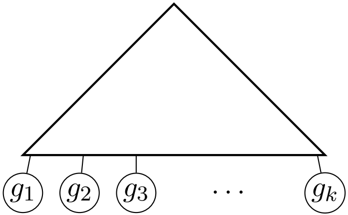

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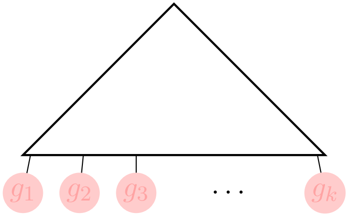

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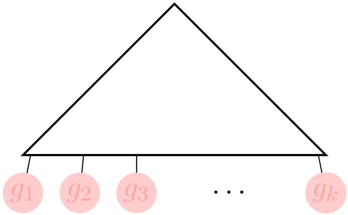

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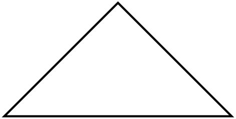

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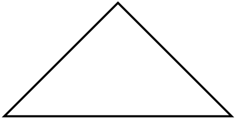

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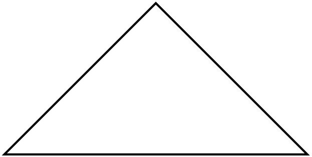

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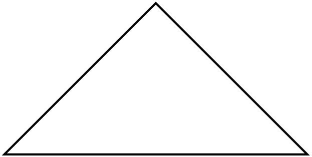

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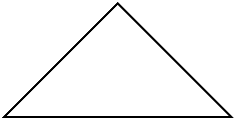

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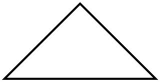

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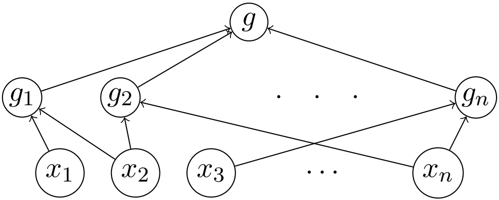

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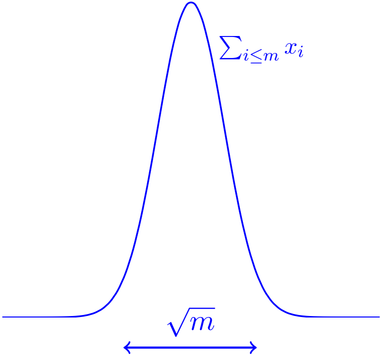

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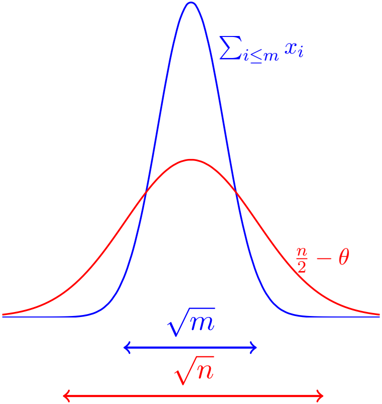

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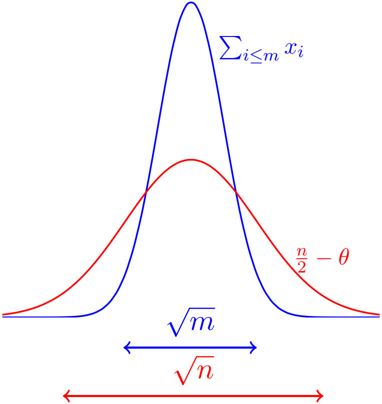

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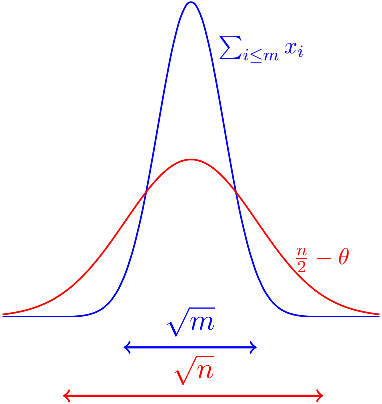

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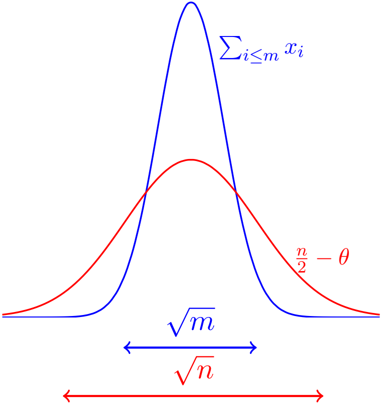

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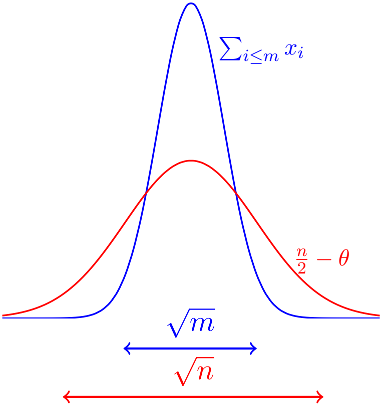

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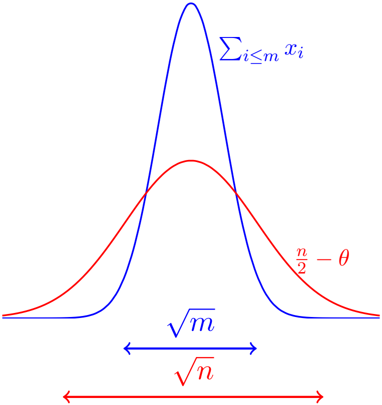

Refining Peres' theorem (contd.)

Lemma (Peres extension)

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- Proof of lemma via standard CLT + critical index argument.

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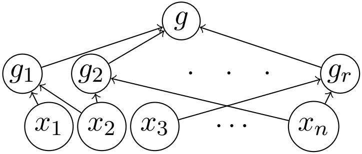

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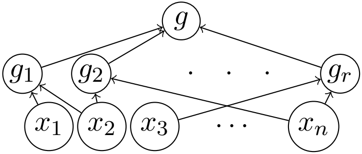

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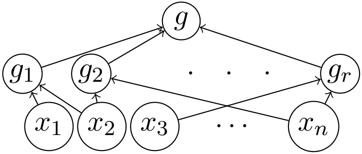

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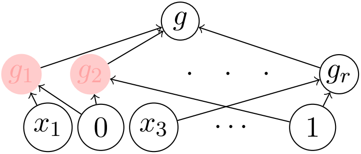
- New Peres: $(1 - p^{0.1})$ gates highly biased.



Back to the wires lower bound

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For some $\delta > 0$*, and* $C \in \mathrm{TC}^0_w(n^{1+\delta^d}, d)$*,* $\mathrm{Corr}(C, \mathrm{PARITY}) = o(1)$*.*

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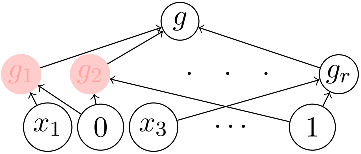

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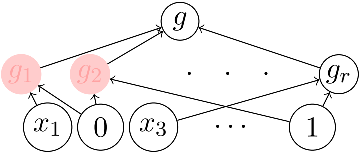

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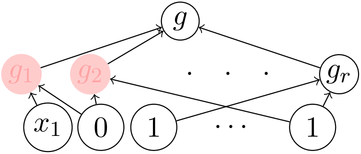

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Thank you