Name: ____________________

Pid: ____________________

1. (10 points) Find $AB$ where $A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 4 \\ -1 & 5 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ 1 & 3 \\ 2 & -4 \end{bmatrix}$.

**Solution:** The answer is

$$\begin{bmatrix} 6 & -7 \\ 10 & -10 \\ -6 & 11 \end{bmatrix}$$

since

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 4 \\ -1 & 5 & -6 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} -7 \\ -10 \\ 11 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 4 \\ -1 & 5 & -6 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ -6 \end{bmatrix},$$

2. (10 points) Let $U : \mathbb{R}^4 \to \mathbb{R}^4$ be a transformation such that

$$U\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & -3 & 2 \\ -2 & 1 & -1 & 2 \\ 4 & 0 & 0 & -1 \\ -2 & 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

Is $U$ one-to-one?

**Solution:** Let us transform the standard matrix of the transformation into echelon form.

$$\begin{bmatrix} 1 & 2 & -3 & 2 \\ -2 & 1 & -1 & 2 \\ 4 & 0 & 0 & -1 \\ -2 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 2 \\ 0 & 5 & -7 & 6 \\ 0 & -8 & 12 & -9 \\ 0 & 5 & -5 & 4 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 2 & -3 & 2 \\ 0 & 5 & -7 & 6 \\ 0 & 0 & 14 & 3 \\ 0 & 0 & 2 & -2 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 2 & -3 & 2 \\ 0 & 5 & -7 & 6 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 14 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 2 \\ 0 & 5 & -7 & 6 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 17 \end{bmatrix}$$

Hence, the transformation is one-to-one.

3. Let $S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 10 \\ 1 \\ 6 \end{bmatrix} \right\}$

(a) (10 points) Is $S$ linearly independent?

**Solution:** Let us consider echelon form of the matrix correspondin to the set.

$$\begin{bmatrix} 1 & 4 & 10 \\ -1 & 5 & 1 \\ 2 & 0 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 10 \\ 0 & 9 & 11 \\ 0 & -8 & 14 \end{bmatrix}$$

The set is linearly independant.

(b) (10 points) Does $S$ span $\mathbb{R}^3$.

**Solution:** The set spans $\mathbb{R}^{\nVDash}$

4. (20 points) Let $T : \mathbb{R}^3 \to \mathbb{R}$ such that

$$T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \begin{bmatrix}4\\0\\4\end{bmatrix}, T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) = \begin{bmatrix}3\\0\\5\end{bmatrix}, \text{ and } T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\1\\3\end{bmatrix}$$

Find the standard matrix for $T$.

**Solution:** The standard matrix of $T$ is $\begin{bmatrix}1 & 2 & 1\\1 & -1 & 0\\3 & 2 & -1\end{bmatrix}$. Indeed,

$$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) - T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}3\\0\\5\end{bmatrix} - \begin{bmatrix}1\\1\\3\end{bmatrix} = \begin{bmatrix}2\\-1\\2\end{bmatrix}$$

and

$$T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) - T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) = \begin{bmatrix}4\\0\\4\end{bmatrix} - \begin{bmatrix}3\\0\\5\end{bmatrix} = \begin{bmatrix}1\\0\\-1\end{bmatrix}$$

5. (20 points) Let $A = \begin{bmatrix}1 & 2 & -3\\2 & 1 & -3\\-1 & 1 & 0\end{bmatrix}$ and $v$ be a vector from $\mathbb{R}^3$. Find all vectors $x \in \mathbb{R}^3$ such that $Av = Ax$.

**Solution:** Note that the solution set of the homogeneous equation $Ax = 0$ is $x_3 \begin{bmatrix}1\\1\\1\end{bmatrix}$. Let us also denote $Av$ by $b$. We need to solve the eqaution $Ax = b$, but we know that $v$ is a solution of this equation. Hence any solution of $Ax = b$ has the following form $v + x_3 \begin{bmatrix}1\\1\\1\end{bmatrix}$.