Name: ______________________

Pid: ______________________

1. Find a basis for each of the following subspaces associated with the matrix $A = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & -2 & 1 & 0 \\ 1 & 3 & -1 & 2 \\ 2 & 1 & 1 & 0 \end{bmatrix}$ if reduced echelon form of this matrix is $\begin{bmatrix} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$:

   (a) (10 points) Col A, the column space of A.

   Solution: The basis of a column solumn space is equal to a set of pivot columns. Hence, the answer is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

   (b) (20 points) Col A^T, the column space of A^T.

   Solution: Note that Col A^T = Row A. Hence, the basis of a column space of A^T is equal to a set of nonzero rows of the reduced echelon form of A. As a result, the answer is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -8 \end{bmatrix} \right\}$

   (c) (30 points) Nul A, the null space of A.

   Solution: It is easy to see that the solution of the equation $Ax = 0$ in parametric vector form is equal to $x_4 \begin{bmatrix} 6 \\ 4 \\ 8 \\ 1 \end{bmatrix}$. Hence, the answer is equal to $\begin{bmatrix} 6 \\ 4 \\ 8 \\ 1 \end{bmatrix}$.

2. (60 points) Let $A = \begin{bmatrix} 1 & 0 & -2 & -1 \\ -2 & 1 & -2 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & -1 & 1 \end{bmatrix}$

(a) Find the inverse of A.

Solution: Let us use the standar algorithm:

$$\begin{bmatrix} 1 & 0 & -2 & -1 & 1 & 0 & 0 & 0 \\ -2 & 1 & -2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & -1 & 0 & 0 & 1 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & -2 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 4 & 3 & -2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & -1 & 0 & 0 & 1 \\ 0 & 0 & 4 & 3 & -2 & -1 & 1 & 0 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & -2 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -5 & 2 & -1 & 1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & \frac{-2}{5} & \frac{1}{5} & \frac{-1}{5} & \frac{4}{5} \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & -2 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{-1}{5} & \frac{-2}{5} & \frac{2}{5} & \frac{-3}{5} \\ 0 & 0 & 0 & 1 & \frac{-2}{5} & \frac{1}{5} & \frac{-1}{5} & \frac{4}{5} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 & \frac{3}{5} & \frac{1}{5} & \frac{-1}{5} & \frac{4}{5} \\ 0 & 1 & -6 & 0 & \frac{6}{5} & \frac{7}{5} & \frac{-2}{5} & \frac{8}{5} \\ 0 & 0 & 1 & 0 & \frac{-1}{5} & \frac{-2}{5} & \frac{2}{5} & \frac{-3}{5} \\ 0 & 0 & 0 & 1 & \frac{-2}{5} & \frac{1}{5} & \frac{-1}{5} & \frac{4}{5} \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{5}{5} & \frac{-3}{5} & \frac{3}{5} & \frac{-2}{5} \\ 0 & 1 & 0 & 0 & 0 & \frac{-5}{5} & \frac{10}{5} & \frac{-10}{5} \\ 0 & 0 & 1 & 0 & \frac{-1}{5} & \frac{-2}{5} & \frac{2}{5} & \frac{-3}{5} \\ 0 & 0 & 0 & 1 & \frac{-2}{5} & \frac{1}{5} & \frac{-1}{5} & \frac{4}{5} \end{bmatrix}$$

Hence, the answer is $\frac{1}{5} \begin{bmatrix} 1 & -3 & 3 & -2 \\ 0 & -5 & 10 & -10 \\ -1 & -2 & 2 & -3 \\ -2 & 1 & -1 & 4 \end{bmatrix}$

(b) Find the determinant of A.

Solution: The answer is 5 since we need only row addition operations and one row exchange operations to get matrix $\begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & -6 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -5 \end{bmatrix}$ from A and the matrix $\begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & -6 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -5 \end{bmatrix}$ has determinant -5.

3. (40 points) Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear map such that

$$T(e_1) = e_2, \qquad T(e_2) = 2e_1, \qquad T(e_3) = he_4, \qquad T(e_4) = e_3.$$

For which values of h, the transformation T is invertible.

Solution: Let us write the standard matrix of T, its equal to $A = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & h & 0 \end{bmatrix}$.

The transformation T is invertible iff A is invertible. However, $\det A = 2h$. As a result, T is invertible iff $h \neq 0$.

4. (a) (15 points) Let S_1 be the set of all vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ such that $x + y^2 = 1$. Is S_1 a vector subspace of $\mathbb{R}^2$? If you answer "No" state at least one vector space condition which is not true.

> **Solution:** It is easy to see that 0 does not belong to this set, since $0 + 0^2 \neq 1$. Hence, it is not a subspace.

(b) (15 points) Let S_2 be the set of all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \cdot \sin x = 0$ for all $x \in \mathbb{R}$. Is S_2 a vector subspace of the space of all functions from $\mathbb{R}$ to $\mathbb{R}$? If you answer "No" state at least one vector space condition which is not true.

> **Solution:** It is a subspace since
>
> - $0 \cdot \sin x = 0$ for all $x \in \mathbb{R}$;
> - if $f(x) \cdot \sin x = 0$ for all $x \in \mathbb{R}$, then it is obvious that $(cf)(x) \cdot \sin x = c \cdot f(x) \cdot \sin x = 0$ for all $x \in \mathbb{R}$;
> - and finally, if $f(x) \cdot \sin x = g(x) \cdot \sin x = 0$ for all $x \in \mathbb{R}$, then $(f + g)(x) \cdot \sin x = f(x) \cdot \sin x + g(x) \cdot \sin x = 0$ for all $x \in \mathbb{R}$.

(c) (15 points) Let S_3 be the set of all polynomials with odd degree. Is S_3 a subspace of the space of polynomials? If you answer "No" state at least one vector space condition which is not true.

> **Solution:** It is not a subspace since $x^3 + x^2$ and $-x^3$ belong to this set but their sum x^2 does not.

(d) Let S_4 be the set of all polynomials p such that the coefficient of each odd degree monomial x^{2d+1} in p is zero. Is S_4 a subspace of the space of polynomials? If you answer "No" state at least one vector space condition which is not true.

> **Solution:** It is a subspace since
>
> - zero polynomial has all coefficients equal to 0;
> - if the coefficient of each odd degree monomial x^{2d+1} in p is zero, then the coefficient of each odd degree monomial x^{2d+1} in cp is also zero;
> - and finally, if the coefficients of each odd degree monomial x^{2d+1} in p and q are zero, then the coefficient of each odd degree monomial x^{2d+1} in $p+q$ is also zero.

5. Find dimensions of the following spaces.

   (a) (20 points) $\mathrm{Span}\{\cos x, \sin x\}$ in the space of all functions from $\mathbb{R}$ to $\mathbb{R}$.

   Solution: In the class we have proved that $\sin x$ and $\cos x$ are linearly independent. As a result $\dim \mathrm{Span}\{\cos x, \sin x\} = 2$

   (b) (20 points) $\mathrm{Span}\{t, t-5, 1\}$ in the space of all polynomials

   Solution: It is easy to see that $t-5 = t+(-5)\cdot 1$. Hence $\mathrm{Span}\{t, t-5, 1\} = \mathrm{Span}\{t, 1\}$. But t and 1 are linearly independent. As a result $\dim \mathrm{Span}\{t, t-5, 1\} = 2$.

6. (60 points) Find any matrix A such that A is a standard matrix of a trnaformation $T : \mathbb{R}^5 \to \mathbb{R}^3$ if you know that kernel of T is equal to $\mathrm{Span}\{e_2, e_3, e_4\}$.

Solution: Since e_2, e_3, and e_4 belong to the kernel of T, we know that $T(e_2) = T(e_3) = T(e_4) = 0$. Hence the second, the third and the fourth columns of A are equal to zero. Additionaly, e_1 and e_5 do not belong to kernel of T hence the first and the last column of A should not be zero. And it is obvious that any matrix satisfying these constraints fits the statement.

Hence the answer is $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

7. (60 points) Suppose that $A = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 & 4 \\ 9 & 9 & 9 \\ 1 & 0 & 2 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 1 & 1 \\ -1 & 0 & 2 \end{bmatrix}$ and $\det A = 3$, $\det B = 9$, and $\det C = 4$. What is $\det A^{-1}B^TA$.

Solution: It is easy to see that $\det A^{-1} = \frac{1}{3}$ since $(\det A)(\det A^{-1}) = \det I = 1$. Additionally, we proved that $\det B^T = \det B = 9$.

Hence, the answer is $\det A^{-1}B^TA = \det A^{-1} \det B^T \det A = \frac{1}{3} \cdot 9 \cdot 3 = 9$.