Math 18 Review 3 December 8, 2017

Name: ____________________

Pid: ____________________

1. Please do not forget to **fill out your name and PID**.
2. During the exam, you can not use calculators and other materials other than a single-side handwritten sheet of paper.
3. Every question in the exam consists of two parts: the part where you need to write the answer and the part where you need to write the justification of your answer.

1. (a) Let $S$ be a set of all the polynomials $p$ such that coefficients $a$ and $b$ of $x$ and $x^2$,respectively, in $p$ satisfy the inequality $a^2 + b^2 \geq 0$. Is $S$ a vector space (answer yes or no)?

   **Solution:** Yes.

   (b) (10 points) Justify your answer

   **Solution:** Note that for any real numbers $a$ and $b$, $a^2 + b^2 \geq 0$ holds. Hence, $S$ is a set of all the polynomials and we know that this set is a vector space.

2. (a) Let $U = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right\}$. Is $U$ a linearly independet set (answer yes or no)?

   **Solution:** Yes.

   (b) (10 points) Justify your answer

   **Solution:** Let us consider a matrix $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$. Note that

   $$\det A = (-1)^{1+3}(-1) \det \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = 1.$$

   Hence, $A$ is invertible and its columns are linearly independent.

3. (a) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. Find all the eigenvalues.

   **Solution:** 2

   (b) (10 points) Justify your answer

   **Solution:** Let us compute the characteristic polynomial of $A$

   $$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 & 1 \\ -1 & 1-\lambda & 1 \\ 1 & 0 & 1-\lambda \end{bmatrix} =$$

   $$(-1)^{1+2} \det \begin{bmatrix} -1 & 1 \\ 1 & 1-\lambda \end{bmatrix} + (-1)^{2+2}(1-\lambda) \det \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}$$

   $$= (1-\lambda) + 1 + (1-\lambda)^3 - (1-\lambda) = 1 + (1-\lambda)^3.$$

   Note that the only solution of this polynomial is $\lambda = 2$. Hence, the only eigenvalue of $A$ is 2.

4. (a) Let $A$ be an $n \times n$ matrix such that $A^2 = 0$. Is it true that the only eigenvalue of $A$ is 0 (answer yes or no)?

**Solution:** Yes.

(b) (10 points) Justify your answer

**Solution:** Note that $A$ is not invertible (since $A^2 = 0$ is not invertible). Hence, 0 is an eigenvalue of $A$.

Let us assume now that $\lambda$ is some eigenvalue of $A$. It means that there is a vector $v \neq 0$ such that $Av = \lambda v$. If we multiply the inequality by $A$ we get $0A^2v = \lambda Av = \lambda^2 v$. Hence, $\lambda^2 v = 0$ and $\lambda = 0$.

5. (a) Let $\mathcal{B} = \{u_1, \ldots, u_n\}$ be an orthonormal basis of $\mathbb{R}^n$ is it true that for every $x \in \mathbb{R}^n$, $||[x]_\mathcal{B}|| = ||x||$ (answer yes or no)?

**Solution:** Yes.

(b) (10 points) Justify your answer

**Solution:** Let us define $c_1, \ldots, c_n$ such that $[x]_\mathcal{B} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$. Note that

$$\begin{aligned} ||x||^2 = \langle x, x\rangle = \langle c_1u_1 + \ldots c_nu_n, x\rangle = \langle c_1u_1 + \ldots c_nu_n, c_1u_1 + \ldots c_nu_n\rangle = \\ c_1c_1\langle u_1, u_1\rangle + c_1c_2\langle u_1, u_2\rangle + \cdots + c_1c_n\langle u_1, u_n\rangle + \cdots + c_nc_n\langle u_n, u_n\rangle = \\ c_1^2\langle u_1, u_1\rangle + \cdots + c_n^2\langle u_n, u_n\rangle = c_1^2 + \cdots + c_n^2 = ||[x]_\mathcal{B}||^2 \end{aligned}$$

6. (a) Let $A$ be a $2 \times 2$ matrix such that $\det A = 6$ and $A$ has two different integer eigenvalues. Is it possible that one of the eigenvales of $A$ is equal to 5 (answer yes or no)?

**Solution:** No.

(b) (10 points) Justify your answer

**Solution:** Since $A$ has two different eigenvalues, $A$ is diagonalizible i.e., there are an invertible matrix $P$ and a diagonal matrix $D$ such that $A = PDP^{-1}$. Note that $\det D$ is equal to the product of eigenvalues and $6 = \det A = \det(PDP^{-1}) = \det P \det D \det P^{-1} = \det D$. But 6 is not divisible by 5.