Systems of Linear Equations

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Institute:
UC San Diego

Grading Policy

All this information is available on the website:
math.ucsd.edu/∼aknop/ucsd/18/2017/

- Homework 5 to 10% of the course grade.

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- The final exam 35% of the course grade.

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Linear Equations

DEFINITION

A **linear equation** in the variables x_1, ..., x_n is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where a_1, ..., a_n, and b are real numbers (i.e. $a_1, \ldots, a_n, b \in \mathbb{R}$).

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EXAMPLE

1. Is $2x_1 + 4x_3 + x_5 = x_6$ a linear equation in the variables x_1, ..., x_6?

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EXAMPLE

1. $2x_1 + 4x_3 + x_5 = x_6$ is a linear equation in the variables x_1, ..., x_6 since it is equivalent to $2x_1 + 0x_2 + 4x_3 + x_5 + (-1)x_6 = 0$;

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2. Is $x_1 + \sqrt{5}x_2 = \sqrt{6}(1 - x_3)$ a linear equation in the variables x_1, x_2, x_3.

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1. $x_1 = x_2x_3 + x_4$ is a linear equation in the variables x_1, x_2;
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Systems of Linear Equations

DEFINITION

A **system of linear equations** in the variables x_1, ..., x_n is a collection of one or more linear equations in the variables x_1, ..., x_n.

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EXAMPLE

$$\begin{aligned} 2x_1 - x_2 + 3x_3 &= 8 \\ x_1 \qquad\quad - 4x_3 &= -7 \end{aligned}$$

Solutions of a System of Linear Equations

DEFINITION

A **solution of of the system** of linear equations in the variables x_1, ..., x_n is a list $(s_1, \ldots, s_n)$ of real numbers such that each equation in the system became true if we substitute the values s_1, ..., s_n to the variables x_1, ..., x_n respectively.

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EXAMPLE

$(5, 6.5, 3)$ is a solution of

$$\begin{aligned} 2x_1 - x_2 + 1.5x_3 &= 8 \\ x_1 \qquad - \quad 4x_3 &= -7 \end{aligned}$$

since $2 \cdot 5 - 6.5 + 1.5 \cdot 3 = 8$ and $5 - 4 \cdot 3 = -7$.

Solutions sets

DEFINITION

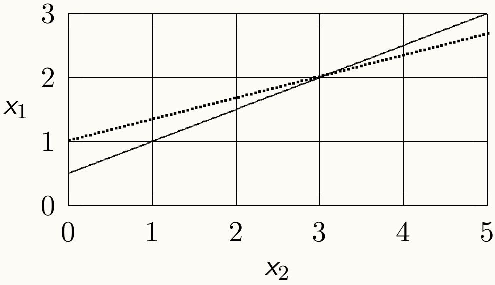
The set of all possible solutions of a system of linear equations is called a **solution set**.

Solutions of a System of Linear Equations

Solution set of the system

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

is $\{(2, 3)\}$.

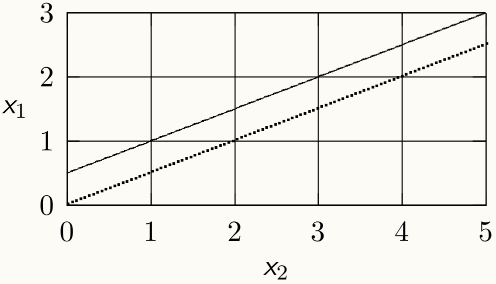


Solutions of a System of Linear Equations

Solution set of the system

$$x_1 - 2x_2 = -1$$
$$x_1 - 2x_2 = 0$$

is empty.

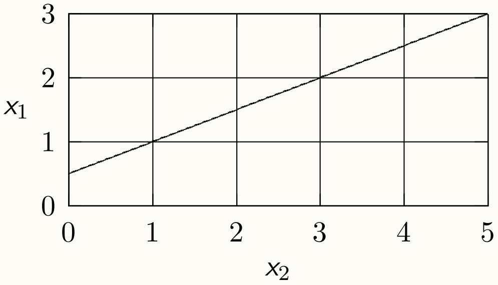


Solutions of a System of Linear Equations

Solution set of the system

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ 2x_1 - 4x_2 &= -2 \end{aligned}$$

is equal to $\{(-1 + 2x_2, x_2) \mid x_2 \in \mathbb{R}\}$.



Solutions of a System of Linear Equations

A system of two linear equations in two variables has

1. no solution, or
2. exactly one solution, or
3. infinitely many solutions.

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Matrix notation

The most important information about a system of equations we can write in a rectangular array.

EXAMPLE

The system

$$\begin{aligned} x_1 - 2x_2 + 4x_3 &= -1 \\ 2x_1 \qquad\quad + \ \ x_3 &= -2 \end{aligned}$$

We may rewrite as

$$\begin{bmatrix} 1 & -2 & 4 & -1 \\ 2 & 0 & 1 & -2 \end{bmatrix}$$

We call such a matrix **augmented matrix** and this matrix without last column is called **coefficient matrix**.

Solving a Linear System

Let us solve the following system.

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

Solving a Linear System

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Add 4 times the first equation to the third equation.

$$\begin{array}{r} 4 \cdot [\text{equation 1:}] \\ +[\text{equation 3:}] \\ \hline [\text{new equation 3:}] \end{array} \qquad \begin{array}{rcr} 4x_1 - 8x_2 + 4x_3 & = & 0 \\ -4x_1 + 5x_2 + 9x_3 & = & -9 \\ \hline -3x_2 + 13x_3 & = & -9 \end{array}$$

Solving a Linear System

$$\begin{aligned} x_1 - 2x_2 + \quad x_3 &= 0 \\ 2x_2 - \ 8x_3 &= 8 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

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Divide equation 2 by 2.

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

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Add 3 times equation 2 to equation 3.

$$\begin{array}{r} 3 \cdot \text{[equation 2:]} \\ +\text{[equation 3:]} \\ \hline \text{[new equation 3:]} \end{array} \qquad \begin{array}{rcr} 3x_2 - 12x_3 & = & 12 \\ -3x_2 + 13x_3 & = & -9 \\ \hline x_3 & = & 3 \end{array}$$

Solving a Linear System

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Solving a Linear System

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Add 2 times equation 2 to equation 1.

$$\begin{array}{r} 2 \cdot \text{[equation 2:]} \\ +\text{[equation 1:]} \\ \hline \text{[new equation 1:]} \end{array} \qquad \begin{array}{crcc} & 2x_2 - 8x_3 & = & 8 \\ x_1 & -2x_2 + x_3 & = & 0 \\ \hline x_1 & -7x_3 & = & 8 \end{array}$$

Solving a Linear System

$$\begin{aligned} x_1 \quad - 7x_3 &= 8 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & -7 & 8 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Solving a Linear System

$$\begin{aligned} x_1 \quad - 7x_3 &= 8 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & -7 & 8 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Add 4 times equation 3 to equation 2.

$4 \cdot$ [equation 3:]	$4x_3$	$=$	12
$+$[equation 1:]	$x_2 - 3x_3$	$=$	4
[new equation 2:]	x_2	$=$	16

Solving a Linear System

$$\begin{aligned} x_1 \qquad - 7x_3 &= 8 \\ x_2 \qquad\quad &= 16 \\ x_3 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & -7 & 8 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Solving a Linear System

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Add 7 times equation 3 to equation 1.

$$\begin{array}{r} 7 \cdot \text{[equation 3:]} \\ +\text{[equation 1:]} \\ \hline \text{[new equation 1:]} \end{array} \qquad \begin{array}{rcl} 7x_3 & = & 21 \\ x_1 - 7x_3 & = & 8 \\ \hline x_1 & = & 29 \end{array}$$

Solving a Linear System

$$\begin{aligned} x_1 \qquad\quad &= 29 \\ x_2 \quad\, &= 16 \\ x_3 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

General Case

ELEMENTARY ROW OPERATIONS

Replacement Replace one row by the sum of itself and multiple of another row.

Interchange Interchange two rows.

Scaling Multiply a row by a nonzero constant.

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Note that we may apply these operations to any matrix. Not only to the augmented matrix of a system of equations.

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Replacement Replace one row by the sum of itself and multiple of another row.

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Note that we may apply these operations to any matrix. Not only to the augmented matrix of a system of equations.

DEFINITION

We call two matrices **row equivalent** if there is a sequence of elementary row operations that transform one matrix into the other.

Conservation of a Solutions Set

THEOREM

If the augmented matrices of two systems are row equivalent, then two systems have the same solutions sets.

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If the augmented matrices of two systems are row equivalent, then two systems have the same solutions sets.

TWO FUNDAMENTAL QUESTIONS ABOUT A LINEAR SYSTEM

1. Is the system consistent; that is, does at least one solution exist?
2. If solution exists, is it the only one?; that is, is the solution unique?

The Fundamental Questions

EXAMPLE

Let us answer this questions about the system

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 5x_1 - 8x_2 + 7x_3 &= 1 \end{aligned}$$

The Fundamental Questions

EXAMPLE

Let us answer this questions about the system

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 5x_1 - 8x_2 + 7x_3 &= 1 \end{aligned}$$

Augmented matrix of this system is

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

The Fundamental Questions

EXAMPLE

Let us answer this questions about the system

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

Let us interchange rows 1 and 2.

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

The Fundamental Questions

EXAMPLE

Let us answer this questions about the system

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

Add $-5/2$ times row 1 to row 3.

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -1/2 & 2 & -3/2 \end{bmatrix}$$

The Fundamental Questions

EXAMPLE

Let us answer this questions about the system

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -1/2 & 2 & -3/2 \end{bmatrix}$$

Add $1/2$ times row 2 to row 3.

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}$$

The Fundamental Questions

EXAMPLE

Let us answer this questions about the system

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}$$

The corresponding system is

$$\begin{aligned} 2x_1 - 3x_2 + 2x_3 &= 1 \\ x_2 - 4x_3 &= 8 \\ +0 &= 5/2 \end{aligned}$$