
Row Reduction and Echelon Forms

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Systems of Linear Equations

EXAMPLE

Let us answer this questions about the system

$$x_1 + x_2 + x_3 = 1$$

$$x_1 + x_2 + 2x_3 = 3$$

$$x_2 + x_3 = 2$$

Systems of Linear Equations

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$$x_1 + x_2 + x_3 = 1$$

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Augmented matrix of this system is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

Systems of Linear Equations

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Let us answer this questions about the system

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

Let add -1 times the row 1 to the row 2.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

Systems of Linear Equations

EXAMPLE

Let us answer this questions about the system

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

Let us exchange the rows 2 and 3.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Systems of Linear Equations

EXAMPLE

Let us answer this questions about the system

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Add -1 times row 3 to row 2.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Systems of Linear Equations

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$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Add -1 times row 3 to row 1.

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Systems of Linear Equations

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Systems of Linear Equations

EXAMPLE

Let us answer this questions about the system

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The corresponding system is

$$x_1 = -1$$

$$x_2 = 0$$

$$x_3 = 2$$

Echelon form

ECHELON FORM

A matrix is in **echelon form** if the following constraints hold.

- 1 All nonzero rows are above any rows of all zeros.
- 2 Each leading entry of a row is in a column to the right of the leading entry of the row above it.
- 3 All entries in a column below a leading entry are zeros.

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Is the following matrix in echelon form?

$$\begin{bmatrix} 1 & -2 & 1 & 0 \end{bmatrix}$$

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The following matrix is in echelon form.

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If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form**.

- ① The leading entry in each nonzero row is 1.
- ② Each leading one is the only nonzero element in its column.

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EXAMPLE

Is the following matrix in reduced echelon form?

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 12 & 0 \end{bmatrix}$$

Echelon form

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EXAMPLE

The following matrix is not in reduced echelon form.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 12 & 0 \end{bmatrix}$$

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If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form**.

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EXAMPLE

The following matrix is in reduced echelon form.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Echelon form

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Each matrix A is row equivalent to one and only one reduced echelon matrix.

We call this matrix **reduced echelon form** of A .

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EXAMPLE

Let us consider the first example from this lecture:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

Echelon form

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Each matrix A is row equivalent to one and only one reduced echelon matrix.

We call this matrix **reduced echelon form** of A .

EXAMPLE

Let us consider the first example from this lecture:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Pivot positions

DEFINITION

A **pivot position** of a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A .

A **pivot column** is a column of A that contains a pivot position.

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Pivot positions

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A **pivot position** of a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A .

A **pivot column** is a column of A that contains a pivot position.

Let us find pivot positions of the following matrix.

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 4 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Pivot positions

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 4 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Let us exchange row 1 and row 4.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 4 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Pivot positions

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 4 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Pivot positions

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 4 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Eliminate all the other elements in the first column.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Pivot positions

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

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Eliminate all the other elements in the first column.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Pivot positions

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Exchange line 3 and line 4.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The Row Reduction Algorithm

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

The Row Reduction Algorithm

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

STEP 1

Begin with leftmost nonzero column. This is a pivot column.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

The Row Reduction Algorithm

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

STEP 2

Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

The Row Reduction Algorithm

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

STEP 3

Use replacement operations to create all zeros in all positions below the pivot.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

The Row Reduction Algorithm

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

STEP 4

Ignore the row containing the pivot and any rows above it. Apply steps 1-3 until there are no more nonzero rows.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

The Row Reduction Algorithm

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

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$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

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$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

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$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

STEP 5

Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by scaling operation.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

The Row Reduction Algorithm

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

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$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

The Row Reduction Algorithm

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Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by scaling operation.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

The Row Reduction Algorithm

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Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by scaling operation.

$$\begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

The Row Reduction Algorithm

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

STEP 5

Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by scaling operation.

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Solutions of Linear Systems

Let us consider the system corresponding to matrix we found in the previous example.

$$x_1 - 2x_3 + 3x_4 = -24$$

$$x_2 - 2x_3 = -7$$

$$x_5 = 4$$

Solutions of Linear Systems

Let us consider the system corresponding to matrix we found in the previous example.

$$\begin{aligned}x_1 - 2x_3 + 3x_4 &= -24 \\x_2 - 2x_3 &= -7 \\x_5 &= 4\end{aligned}$$

In other words, the solution set may be described in the following manner.

$$\begin{aligned}x_1 &= -24 + 2x_3 - 3x_4 \\x_2 &= -7 + 2x_3 \\x_5 &= 4 \\x_3 &\text{ is free} \\x_4 &\text{ is free}\end{aligned}$$

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Note that for answering about existence and uniqueness of a solution we do not need reduced echelon form. Echelon form is just enough.

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Note that for answering about existence and uniqueness of a solution we do not need reduced echelon form. Echelon form is just enough.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

It is already easy to see that there are more than one solution.