Vector Equations

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Previously On Math 18

We considered the system

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}.$$

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We transformed it into the matrix

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -5 \\ 0 & 1 & -2 & 0 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}.$$

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With the corresponding system

$$\begin{aligned} x_1 \quad - 2x_3 + 3x_4 \quad &= -24 \\ x_2 - 2x_3 \qquad\qquad &= -7 \\ x_5 &= 4 \end{aligned}$$

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In other words, the solution set may be described in the following manner.

$$\begin{aligned} &x_1 = -24 + 2x_3 - 3x_4 \\ &x_2 = -7 + 2x_3 \\ &x_5 = 4 \\ &x_3 \text{ is free} \\ &x_4 \text{ is free} \end{aligned}$$

Existence and Uniqueness of a Solution

THEOREM

- *A linear system is consistent iff the rightmost column of the augmented matrix is not a pivot column.*

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- *A linear system is consistent iff the rightmost column of the augmented matrix is not a pivot column.*
- *If a linear system is consistent, then the solution set contains only unique solution when the there are no free variables, or infinitely many solutions when there is at least one free variable.*

Existence of a Solution

Let us consider the matrix

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & 1 & 0 & 3 \\ 3 & 3 & 1 & 4 \end{bmatrix}$$

Existence of a Solution

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & 1 & 0 & 3 \\ 3 & 3 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} \color{orange}{1} & 2 & 1 & 4 \\ 2 & 1 & 0 & 3 \\ 3 & 3 & 1 & 4 \end{bmatrix}$$

Existence of a Solution

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & 1 & 0 & 3 \\ 3 & 3 & 1 & 4 \end{bmatrix}$$

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Existence of a Solution

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The system of equation is inconsistent.

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & 1 & 0 & 3 \\ 3 & 3 & 1 & 4 \end{bmatrix}$$

Uniqueness of a Solution

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$$\begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & -3 & 0 & -3 & -11 \\ 0 & 0 & 1 & -1 & -3 \end{bmatrix}$$

Uniqueness of a Solution

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & -3 & 0 & -3 & -11 \\ 0 & 0 & 1 & -1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -2 & -19/3 \\ 0 & 1 & 0 & 1 & 11/3 \\ 0 & 0 & 1 & -1 & -3 \end{bmatrix}$$

Uniqueness of a Solution

$$\begin{bmatrix} 1 & 0 & 0 & -2 & -19/3 \\ 0 & 1 & 0 & 1 & 11/3 \\ 0 & 0 & 1 & -1 & -3 \end{bmatrix}$$

$x_1 = -19/3 + 2x_4$
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DEFINITION

A matrix with only one column is called a **vector** or **column vector**.

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EXAMPLE

$$u = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad v = \begin{bmatrix} .1 \\ .2 \end{bmatrix} \quad w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

where w_1 and w_2 are any real numbers.

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The set of all vectors with two entries is called $\mathbb{R}^2$.

Equal Vectors

DEFINITIONS

Two vectors are equal iff their corresponding entries are equal.

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EXAMPLE

Are vectors $\begin{bmatrix} 7 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 7 \end{bmatrix}$ equal?

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Vectors $\begin{bmatrix} 7 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 7 \end{bmatrix}$ are not equal.

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Are vectors $\begin{bmatrix} 44 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \cdot 22 \\ 2 - 1 \end{bmatrix}$ equal?

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Given two vectors u and v their **sum** is obtained by adding corresponding entries of u and v.

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EXAMPLE

$$\begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+7 \\ 4+1 \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \end{bmatrix}$$

Scalar Multiplies of Vectors

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Given a vector u and a real number c, the **scalar multiple** of u by c is a vector cu obtained by multiplying of each entry of u by c.

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$$\text{if } u = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \text{ and } c = 5, \text{ then } cu = \begin{bmatrix} 5 \cdot 2 \\ 5 \cdot 3 \end{bmatrix} = \begin{bmatrix} 10 \\ 15 \end{bmatrix}.$$

Vectors in $\mathbb{R}^n$

DEFINITION

If $n > 0$, then $\mathbb{R}^n$ denotes a set of all vectors with n entries.
We denote by 0 the vector consisting of all zeros.
Equality, addition and multiplication by scalar we define entry by entry as in $\mathbb{R}^2$.

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ALGEBRAIC PROPERTIES OF $\mathbb{R}^N$

1. $u + v = v + u$
2. $(u + v) + w = u + (v + w)$
3. $u + 0 = 0 + u = u$
4. $u + (-u) = -u + u = 0$ where $-u$ denotes $(-1)u$
5. $c(u + v) = cu + cv$
6. $(c + d)u = cu + du$
7. $c(du) = (cd)u$
8. $1u = u$

Linear combinations

DEFINITION

Given vectors $u_1, \dots, u_l \in \mathbb{R}^n$ and real numbers c_1, ..., c_l. The vector equal to

$$v = c_1 u_1 + \cdots + c_l u_l$$

is called a **linear combination** of u_1, ..., u_l.

Linear combinations

Let us consider vectors

$$a = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}, \text{ and } c = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}.$$

Does c can be written as a linear combination of a and b?

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Are there real x and y such that $xa + yb = c$.

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$$\begin{bmatrix} x + 2y \\ -2x + 5y \\ -5x + 6y \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

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Are there real x and y such that

$$\begin{aligned} x + 2y &= 7 \\ -2x + 5y &= 4 \\ -5x + 6y &= -3. \end{aligned}$$

Vector equations

THEOREM

A vector equation

$$x_1 a_1 + \ldots x_l a_l = b$$

has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} a_1 & a_2 & \ldots & a_l & b \end{bmatrix}$$

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REMARK

Asking if $b \in \mathrm{Span}\{a_1, \ldots, a_l\}$ is equivalent to asking if an equation $x_1 a_1 + \cdots + x_l a_l = b$ has a solution.

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2. If $u, v \in \mathbb{R}^3$ and v is not multiple of u, then $\text{Span}\{u, v\}$ represents a plane.

Linear Combinations in Applications

EXAMPLE

A company manufactures two products. For $1\$$ worth of product B, the company spends $.45$ on materials, $.25$ on labor, and $.15$ on overhead. For $1\$$ worth of product C, the company spends $.40$ on materials, $.30$ on labor, and $.20$ on overhead.

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A vector $100b$ lists the various costs for producing the product B $100\$$ worth.

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2. Suppose the company wishes to manufacture x_1 dollars worth of product B and x_2 dollars worth of product C. Give a vector that describes the various costs the company will have.

A vector $100b$ lists the various costs for producing the product B $100\$$ worth. The costs of manufacturing x_1 dollars worth of B are given by vector $x_1 b$ and the costs of manufacturing x_2 dollars worth of A are given by vector $x_2 c$. Hence the total costs are $x_1 b + x_2 c$.

A Product of a Matrix and a Column

DEFINITION

If A is a $n \times m$ matrix, with columns a_1, ..., a_m and $x \in \mathbb{R}^n$, then the product of A and x denoted as Ax is

$$Ax = x_1 a_1 + \cdots + x_n a_n.$$

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EXAMPLE

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + 7 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Solutions of Matrix Equations

THEOREM

If A is a matrix with columns $a_1, \dots, a_n \in \mathbb{R}^m$, then the matrix equation

$$Ax = b$$

has the same solution set as the vector equation

$$x_1 a_1 + \dots + x_n a_n = b$$

which in turn has the same solution as a system of linear equations with augmented matrix is

$$\begin{bmatrix} a_1 & a_2 & \dots & a_n & b \end{bmatrix}.$$

Existence of Solutions

THEOREM

The equation $Ax = b$ has a solution iff b is a linear combination of the columns of A.