Matrix equation $\mathbf{Ax} = \mathbf{b}$

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Previously On Math 18

DEFINITION

If $v_1, \ldots, v_\ell \in \mathbb{R}^n$, then a set of all linear combinations of them is called $\text{Span}\{v_1, \ldots, v_\ell\}$.

Let $u \in \mathbb{R}^2$ be a nonzero vector.

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Let $u \in \mathbb{R}^2$ be a nonzero vector. Then $\mathrm{Span}\{u\}$ is a line.

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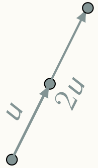

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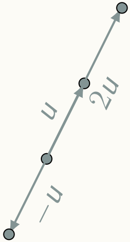

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Note that, the set $\text{Span}\{a_1, \ldots, a_\ell\}$ consists all the following vectors $x_1 a_1 + \cdots + x_\ell a_\ell$.

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REMARK

Asking if $b \in \text{Span}\{a_1, \ldots, a_\ell\}$ is equivalent to asking if an equation $x_1 a_1 + \cdots + x_\ell a_\ell = b$ has a solution.

A Product of a Matrix and a Column

DEFINITION

If A is a $n \times m$ matrix, with columns a_1, ..., a_m and $x \in \mathbb{R}^n$, then the product of A and x denoted as Ax is

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EXAMPLE

$$\begin{bmatrix} 1 & 1 & -1 & 4 \\ 0 & -2 & 3 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \\ 1 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 1 & -1 & 4 \\ 0 & -2 & 3 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \\ 1 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + 7 \begin{bmatrix} -1 \\ 3 \end{bmatrix} + 1 \begin{bmatrix} 42 \end{bmatrix}$$

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Solutions of Matrix Equations

THEOREM

If A is a matrix with columns $a_1, \ldots, a_n \in \mathbb{R}^m$, then the matrix equation

$$Ax = b$$

has the same solution set as the vector equation

$$x_1 a_1 + \cdots + x_n a_n = b$$

which in turn has the same solution as a system of linear equations with augmented matrix is

$$\begin{bmatrix} a_1 & a_2 & \ldots & a_n & b \end{bmatrix}.$$

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which in turn has the same solution as a system of linear equations with augmented matrix is

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Existence of Solutions

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The equation $Ax = b$ has a solution iff b is a linear combination of the columns of A.

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Let us row reduce the corresponded augmented matrix.

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 7 & 5 & b_3 + 3b_1 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 0 & 0 & b_3 + 3b_1 - \frac{1}{2}(b_2 + 4b_1) \end{bmatrix}$$

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Hence, not for any b there is a solution.

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THEOREM

Let A be a $m \times n$ matrix. Then the following statements are logically equivalent.

1. *For each $b \in \mathbb{R}^m$, the equation $Ax = b$ has a solution.*
2. *Each $b \in \mathbb{R}^m$ is a linear combination of the columns of A.*
3. *The columns of A span $\mathbb{R}^m$ ($\text{Span}\{a_1, \ldots, a_n\} = \mathbb{R}^m$ where a_1, ..., a_n are columns of A).*
4. *A has a pivot position in every row.*

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If property 4 false, then the last column of U is a zero column. Let d ba a column with 1 in the last entry. Then $\begin{bmatrix} U & d \end{bmatrix}$ is inconsistent. Transform this system back to $Ax = b$, this system is also inconsistent. □