Solution sets

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Computation of Ax

EXAMPLE

Let us compute Ax where

$$A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix} \text{ and } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

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By the definition

$$\begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} x_3 \begin{bmatrix} 4 \\ -3 \\ 8 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ -x_1 + 5x_2 - 3x_3 \\ 6x_1 - 2x_2 + 8x_3 \end{bmatrix}$$

Computation of Ax

ROW-VECTOR RULE FOR COMPUTING AX

If the product Ax is defined, then the ith entry in Ax is the sum of the products of corresponding entries from row i of A and from the vector x.

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Let us compute

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

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EXAMPLE

Let us compute

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 2 \cdot 2 + 3 \cdot (-1) \\ 0 \cdot 1 + 5 \cdot 2 + (-1) \cdot (-1) \end{bmatrix}$$

Computation of *Ax*

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Let us compute

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 2 \cdot 2 + 3 \cdot (-1) \\ 0 \cdot 1 + 5 \cdot 2 + (-1) \cdot (-1) \end{bmatrix} = \begin{bmatrix} 2 \\ 11 \end{bmatrix}$$

Properties of Matrix-Vector Product

THEOREM

If A is an $m \times n$ matrix, $u, v \in \mathbb{R}^n$, and c is a scalar, then:

1. $A(u + v) = Au + Av$;
2. $A(cu) = c(Au)$.

Properties of Matrix-Vector Product

PROOF.

Let us consider case when $n = 3$.

$$A(u+v) = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix} =$$

$$(u_1 + v_1)a_1 + (u_2 + v_2)a_2 + (u_3 + v_3)a_3 =$$

$$(u_1 a_1 + u_2 a_2 + u_3 a_3) + (v_1 a_1 + v_2 a_2 + v_3 a_3) = uA + vA$$

□

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PROOF.

Let us consider case when $n = 3$.

$$A(cu) = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix} = (cu_1)a_1 + (cu_2)a_2 + (cu_3)a_3 = c(u_1 a_1) + c(u_2 a_2) + c(u_3 a_3) = c(Au)$$

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Homogeneous Linear Systems

DEFINITION

The system of linear equations is called **homogeneous** iff it can be rewritten in the form $Ax = 0$.
Solution of homogeneous system is **nontrivial** iff it is not equal to zero vector.

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EXAMPLE

The solution $(1, 2, 1)$ of the system

$$\begin{aligned} x_1 + x_2 + x_3 &= 4 \\ x_1 \qquad - x_3 &= 0 \end{aligned}$$

is nontrivial.

Homogeneous Linear Systems

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The system of linear equations is called **homogeneous** iff it can be rewritten in the form $Ax = 0$.
Solution of homogeneous system is **nontrivial** iff it is not equal to zero vector.

THEOREM

The homogeneous system has nontrivial solution iff the system has at least one free variable.

Homogeneous Linear Systems

EXAMPLE

Let us determine if the following system has nontrivial solution.

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

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Let us consider the augmented matrix.

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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In the reduced echelon form this system equals

$$\begin{bmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 \quad - \tfrac{4}{3}x_3 \quad &= 0 \\ x_2 \quad\quad\quad &= 0 \\ +0 &= 0 \end{aligned}$$

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Solution of this system is

$$x = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix}$$

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Solution of this system is

$$x = x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} = x_3 v$$

Nonhomogeneous Linear Systems

EXAMPLE

Let us solve the following system.

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 7 \\ -3x_1 - 2x_2 + 4x_3 &= -1 \\ 6x_1 + x_2 - 8x_3 &= -4 \end{aligned}$$

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Using row operations we obtain reduced echelon form.

$$\begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{aligned} x_1 \qquad - \tfrac{4}{3}x_3 \qquad &= -1 \\ x_2 \qquad\qquad\quad &= 2 \\ +\,0 &= 0 \end{aligned}$$

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As a vector general solution looks like:

$$x = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} = p + x_3 v$$

Structure of Solution Sets

THEOREM

Let $Ax = b$ be a consistent linear system and and let p be a solution. Then the solution set of $Ax = b$ is the set of all vectors of the form $w = p + v_h$ where is any solution of the homogeneous system $Ax = 0$.

Structure of Solution Sets

WRITING A SOLUTION SET IN PARAMETRIC VECTOR FORM

1. Row reduce augmented matrix to reduced echelon form.

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1. Row reduce augmented matrix to reduced echelon form.
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3. Write a typical solution x as vector entries depends on free variables, if any.
4. Decompose x into linear combination of vectors (with numeric entries) using the free variables as parameters.

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Let us consider a matrix

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 3 & 0 & 3 \end{bmatrix}$$

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Let us consider a matrix

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Transform it into reduced echelon form

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & -3 & 3 \end{bmatrix}$$

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Transform it into reduced echelon form

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$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 + x_3 \\ 1 - x_3 \\ x_3 \end{bmatrix}$$

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$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 + x_3 \\ 1 - x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$$