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# Linear independence

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# Economics

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## EQUILIBRIUM PRICES

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- 4 Steel sector is divided as follows: 60% to Coal, 20% to Electric, and 20% to Steel.

We want to find prices that can be assigned to the total output of various sectors in such a way that the income of each sector exactly balances its expenses.

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# Economics

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## EQUILIBRIUM PRICES

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If we write this data in the table we get.

Purchased by	Coal	Electric	Steel
Coal	.0	.4	.6
Electric	.6	.1	.2
Steel	.4	.5	.2

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Let  $p_C$ ,  $p_E$ , and  $p_S$  be prices for Coal, Electric, and Steel respectively. The statement that prices are equilibrium prices may be written in the following form.

$$.4p_E + .6p_S = p_C$$

$$.6p_C + .1p_E + .2p_S = p_E$$

$$.4p_C + .5p_E + .2p_S = p_S$$

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## EQUILIBRIUM PRICES

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Move all the unknowns to the left.

$$.4p_E + .6p_S = 0$$

$$.6p_C - .9p_E + .2p_S = 0$$

$$.4p_C + .5p_E - .8p_S = 0$$

# Economics

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$$.4p_C + .5p_E - .8p_S = 0$$

Let us transform to row reduced form.

$$\begin{bmatrix} -1 & .4 & .6 & 0 \\ .6 & -0.9 & .2 & 0 \\ .4 & .5 & -.8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -.4 & -.6 & 0 \\ 0 & -0.66 & .56 & 0 \\ 0 & 0.66 & -.56 & 0 \end{bmatrix} \sim$$
$$\begin{bmatrix} 1 & -.4 & -.6 & 0 \\ 0 & 1 & -.85 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -.94 & 0 \\ 0 & 1 & -.85 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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# Economics

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## EQUILIBRIUM PRICES

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$$p = \begin{bmatrix} p_C \\ p_E \\ p_S \end{bmatrix} = \begin{bmatrix} .94p_S \\ .85p_S \\ p_S \end{bmatrix} = p_S \begin{bmatrix} .94 \\ .85 \\ 1 \end{bmatrix}$$

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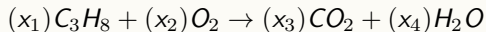
# Balancing Chemical Equations

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## EXAMPLE

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Let us consider the chemical equation



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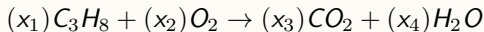
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We want to balance this equation i.e. we want that number of Carbon atoms at the left is equal to number of Carbon atoms at the right etc.

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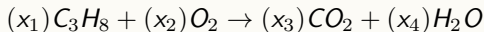
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## EXAMPLE

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If we write it as a system of linear equations we get the following.

$$3x_1 - x_3 = 0$$

$$8x_1 - 2x_4 = 0$$

$$2x_2 - 2x_3 - x_4 = 0$$

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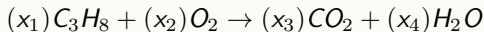
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## EXAMPLE

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Let us consider the chemical equation



If we solve a system we get the following solution.

$$x_1 = \frac{1}{4}x_4, x_2 = \frac{5}{4}x_4, x_3 = \frac{3}{4}x_4, \text{ with } x_4 \text{ free}$$



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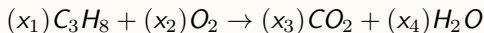
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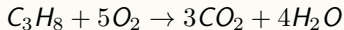
## EXAMPLE

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Let us consider the chemical equation



Hence, the balanced reaction is



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# Linear Independent Sets

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## DEFINITION

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A set of vectors  $\{v_1, \dots, v_p\} \subset \mathbb{R}^n$  is said to be **linear independent**, if the vector equation

$$x_1 v_1 + \dots + x_p v_p = 0$$

has only the trivial solution.

The set of vectors  $\{v_1, \dots, v_p\} \subset \mathbb{R}^n$  is said to be **linear dependent** with weights  $c_1, \dots, c_p$  if

$$c_1 v_1 + \dots + c_p v_p = 0$$

and not all  $c_1, \dots, c_p$  are zeros.

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# Linear Independent Sets

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## EXAMPLE

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Let  $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ ,  $v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ , and  $v_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ .

Let us find if they are independent.

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Transform to row reduced form.

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Thus  $x_1 = 2x_3$  and  $x_2 = -x_3$ . As a result  $2v_1 - v_2 + v_3 = 0$ .

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# Linear Independence of Matrix Columns

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## THEOREM

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*The columns of a matrix  $A$  are linearly independent iff the equation  $Ax = 0$  has only the trivial solution.*

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## EXAMPLE

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Let us check if columns of the matrix  $A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$  are linearly independent.



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# Linear Independence of Matrix Columns

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## THEOREM

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## EXAMPLE

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Study augmented matrix of the equation  $Ax = 0$ .

$$\begin{bmatrix} 0 & 1 & 4 & 0 \\ 1 & 2 & -1 & 0 \\ 5 & 8 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & -2 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 13 & 0 \end{bmatrix}$$

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# Linear Independent Sets

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## EXAMPLE

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Are the following vectors linearly independent?

$$v_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

Note that  $2v_1 = v_2$ , hence,  $-2v_1 + v_2 = 0$ .

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Suppose  $cv_1 + dv_2 = 0$  for  $c, d \neq 0$

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Suppose  $cv_1 + dv_2 = 0$  for  $c, d \neq 0$ , it means that  $v_1 = -\frac{c}{d}v_2$ .

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# Linearly Independent Sets

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## THEOREM

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*A set  $S = \{v_1, \dots, v_p\}$  of two or more vectors is linearly dependent if and only if at least one of the vectors in  $S$  is linearly combination of the others.*

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# Linearly Independent Sets

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## THEOREM

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*If a set contains more vectors than there are entries in each vector, then the set is linearly dependent.*

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## PROOF.

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Let  $A = [a_1 \ \dots \ a_p]$  and  $p > n$ .

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As a result columns of  $A$  are linearly dependant. □

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# Transformations

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## DEFINITION

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A **transformation (or function)**  $T$  from  $\mathbb{R}^n$  to  $\mathbb{R}^m$  is a rule that assigns to each vector  $x \in \mathbb{R}^n$  a vector  $T(x) \in \mathbb{R}^m$ .

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- 2 The notion  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  states that domain of  $T$  is  $\mathbb{R}^n$  and codomain is  $\mathbb{R}^m$ .

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- 3 For vector  $x \in \mathbb{R}^n$  the vector  $T(x)$  is called **image** of  $x$ .
- 4 The set of all images is called the **range** of  $T$ .

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# Matrix Transformations

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## DEFINITION

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If for each  $x \in \mathbb{R}^n$ ,  $T(x)$  is computed as  $Ax$  where  $A$  is  $n \times m$  matrix we call such a transformation as **matrix transformation**.

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# Matrix Transformations

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For simplicity we sometimes denote such a matrix transformation as  $x \mapsto Ax$ .