Linear independence

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Economics

EQUILIBRIUM PRICES

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We want to find prices that can be assigned to the total output of various sectors in such a way that the income of each sector exactly balances its expenses.

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If we write this data in the table we get.

Purchased by	Coal	Electric	Steel
Coal	.0	.4	.6
Electric	.6	.1	.2
Steel	.4	.5	.2

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Let p_C, p_E, and p_S be prices for Coal, Electric, and Steel respectively. The statement that prices are equilibrium prices may be written in the following form.

$$\begin{aligned} .4p_E + .6p_S &= p_C \\ .6p_C + .1p_E + .2p_S &= p_E \\ .4p_C + .5p_E + .2p_S &= p_S \end{aligned}$$

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Move all the unknowns to the left.

$$\begin{aligned} .4p_E + .6p_S &= 0 \\ .6p_C - .9p_E + .2p_S &= 0 \\ .4p_C + .5p_E - .8p_S &= 0 \end{aligned}$$

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Let us transform to row reduced form.

$$\begin{bmatrix} -1 & .4 & .6 & 0 \\ .6 & -0.9 & .2 & 0 \\ .4 & .5 & .-8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -.4 & -.6 & 0 \\ 0 & -0.66 & .56 & 0 \\ 0 & 0.66 & -.56 & 0 \end{bmatrix} \sim$$
$$\begin{bmatrix} 1 & -.4 & -.6 & 0 \\ 0 & 1 & -.85 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -.94 & 0 \\ 0 & 1 & -.85 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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$$p = \begin{bmatrix} p_C \\ p_E \\ p_S \end{bmatrix} = \begin{bmatrix} .94p_S \\ .85p_S \\ p_S \end{bmatrix} = p_S \begin{bmatrix} .94 \\ .85 \\ 1 \end{bmatrix}$$

Balancing Chemical Equations

EXAMPLE

Let us consider the chemical equation

$$(x_1)C_3H_8 + (x_2)O_2 \to (x_3)CO_2 + (x_4)H_2O$$

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We want to balance this equation i.e. we want that number of Carbon atoms at the left is equal to number of Carbon atoms at the right etc.

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$$(x_1)C_3H_8 + (x_2)O_2 \to (x_3)CO_2 + (x_4)H_2O$$

If we write it as a system of linear equations we get the following.

$$\begin{aligned} 3x_1 \quad\quad - \quad x_3 \quad\quad\quad &= 0 \\ 8x_1 \quad\quad\quad\quad\quad - 2x_4 &= 0 \\ 2x_2 - 2x_3 - \quad x_4 &= 0 \end{aligned}$$

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If we solve a system we get the following solution.

$$x_1 = \frac{1}{4}x_4, x_2 = \frac{5}{4}x_4, x_3 = \frac{3}{4}x_4, \text{ with } x_4 \text{ free}$$

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Let us consider the chemical equation

$$(x_1)C_3H_8 + (x_2)O_2 \to (x_3)CO_2 + (x_4)H_2O$$

Hence, the balanced reaction is

$$C_3H_8 + 5O_2 \to 3CO_2 + 4H_2O$$

Linear Independent Sets

DEFINITION

A set of vectors $\{v_1, \ldots, v_p\} \subset \mathbb{R}^n$ is said to be **linear independent**, if the vector equation

$$x_1 v_1 + \cdots + x_p v_p = 0$$

has only the trivial solution.
The set of vectors $\{v_1, \ldots, v_p\} \subset \mathbb{R}^n$ is said to be **linear dependent** with weights c_1, ..., c_p if

$$c_1 v_1 + \cdots + c_p v_p = 0$$

and not all c_1, ..., c_p are zeros.

Linear Independent Sets

EXAMPLE

Let $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.
Let us find if they are independent.

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Transform to row reduced form.

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Thus $x_1 = 2x_3$ and $x_2 = -x_3$. As a result $2v_1 - v_2 + v_3 = 0$.

Linear Independence of Matrix Columns

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The columns of a matrix A are linearly independent iff the equation $Ax = 0$ has only the trivial solution.

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EXAMPLE

Let us check if columns of the matrix $A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$ are linearly independent.

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The columns of a matrix A are linearly independent iff the equation $Ax = 0$ has only the trivial solution.

EXAMPLE

Study augmented matrix of the equation $Ax = 0$.

$$\begin{bmatrix} 0 & 1 & 4 & 0 \\ 1 & 2 & -1 & 0 \\ 5 & 8 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & -2 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 13 & 0 \end{bmatrix}$$

Linear Independent Sets

EXAMPLE

Are the following vectors linearly independent?

$$v_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \; v_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

Note that $2v_1 = v_2$, hence, $-2v_1 + v_2 = 0$.

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Suppose $cv_1 + dv_2 = 0$ for $c, d \neq 0$, it means that $v_1 = -\frac{c}{d} v_2$.

Linearly Independent Sets

THEOREM

A set $S = \{v_1, \ldots, v_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is linearly combination of the others.

Linearly Independent Sets

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If a set contains more vectors than there are entries in each vector, then the set is linearly dependent.

PROOF.

Let $A = \begin{bmatrix} a_1 & \dots & a_p \end{bmatrix}$ and $p > n$.

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As a result columns of A are linearly dependant. □

Transformations

DEFINITION

A **transformation (or function)** T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $x \in \mathbb{R}^n$ a vector $T(x) \in \mathbb{R}^m$.

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4. The set of all images is called the **range** of T.

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For simplicity we sometimes denote such a matrix transformation as $x \mapsto Ax$.