# Linear Transformations

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# Transformations

## DEFINITION

A **transformation (or function)** $T$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $x \in \mathbb{R}^n$ a vector $T(x) \in \mathbb{R}^m$.

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4. The set of all images is called the **range** of $T$.

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**EXAMPLE**

Let us consider transformation that assigns $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ to $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$.

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Let us consider transformation that assigns $\begin{bmatrix}\sin x\end{bmatrix}$ to $\begin{bmatrix}x\end{bmatrix}$.
Domain is $\mathbb{R}^1$ and codomain is $\mathbb{R}^1$.
In other words $T : \mathbb{R}^1 \to \mathbb{R}^1$.

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Let us consider transformation that assigns $\left[\sin x\right]$ to $\left[x\right]$.
Domain is $\mathbb{R}^1$ and codomain is $\mathbb{R}^1$.
In other words $T : \mathbb{R}^1 \to \mathbb{R}^1$.
The range is $[-1, 1]$.

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Let us consider transformation that assigns $\begin{bmatrix} x_1 + 1 \\ x_2 + 2 \\ x_3 + 3 \\ x_4 + 4 \end{bmatrix}$ to $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$.

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Domain is $\mathbb{R}^4$ and codomain is $\mathbb{R}^4$.
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The range is $\mathbb{R}^4$.

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Let us consider transformation that assigns $x_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

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Domain is $\mathbb{R}^2$ and codomain is $\mathbb{R}^3$.
In other words $T\colon \mathbb{R}^2 \to \mathbb{R}^3$.
Note that $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

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If for each $x \in \mathbb{R}^n$, $T(x)$ is computed as $Ax$ where $A$ is $n \times m$ matrix we call such a transformation as **matrix transformation**.

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Note that range of $Ax$ is a set of linear combinations of columns of $A$

# Matrix Transformations

## EXAMPLE

Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$, and $T(x) = Ax$.

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Compute $T(u) = Au = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$

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Solve $\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$

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2. find $x$ such that $Ax = b$;

Solve $\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1.5 \\ 0 & 1 & -.5 \\ 0 & 0 & 0 \end{bmatrix}$

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Hence $x_1 = 1.5$ and $x_2 = -.5$.

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3. is there more than more than one such $x$;

It is easy to see that answer is no.

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1. Every matrix transformation is linear transformation.
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1. Every matrix transformation is linear transformation.
2. If $T$ is linear transformation, then $T(0) = 0$.
3. If $T$ is linear transformation, then $T(cu + dv) = cT(u) + dT(v)$.

# The Matrix of a Linear Transformation

The columns of $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

**EXAMPLE**

Suppose $T(e_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix}$ and $T(e_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$.

Find a formula for image of $x \in \mathbb{R}^2$.

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Note that $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1 e_1 + x_2 e_2$.

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Since $T$ is linear $T(x) = x_1 T(e_1) + x_2 T(e_2) = \begin{bmatrix} 5x_1 - 3x_2 \\ -7x_1 + 8x_2 \\ 2x_1 \end{bmatrix}$.

# The Matrix of a Linear Transformation

**THEOREM**

*Let* $T : \mathbb{R}^n \to \mathbb{R}^m$ *be a linear transformation. Then there is a unique matrix* $A$ *such that* $T(x) = Ax$ *for all* $x \in \mathbb{R}^n$

# Existence and Uniqueness Questions

**DEFINITION**

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $b \in \mathbb{R}^m$ is the image of at least one $x \in \mathbb{R}^n$.

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**DEFINITION**

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **one-to-one** if each $b \in \mathbb{R}^m$ is an image of *at most* one $x \in \mathbb{R}^n$.

# Uniqueness Question

**THEOREM**

*Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then $T$ is one-to-one iff the equation $T(x) = 0$ has only one solution.*

**PROOF.**

Since $T$ is linear, $T(0) = 0$ and we know that $T(x) = 0$ has at most one solution and hence only the trivial solution.

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If $T$ is *not* one-to-one, then there are a $b \in \mathbb{R}^m$ and $u \neq v \in \mathbb{R}^n$ such that $T(u) = T(v) = b$.

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**PROOF.**

Since $T$ is linear, $T(0) = 0$ and we know that $T(x) = 0$ has at most one solution and hence only the trivial solution.
If $T$ is *not* one-to-one, then there are a $b \in \mathbb{R}^m$ and $u \neq v \in \mathbb{R}^n$ such that $T(u) = T(v) = b$.
But since $T$ is linear $T(u - v) = T(u) - T(v) = 0$. But $u - v \neq 0$ hence $T(x) = 0$ has more than one solution, contradiction. $\square$

# Existence Question

**THEOREM**

*Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let $A$ be the standard matrix for $T$. Then:*

1. *$T$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ iff the columns of $A$ spans $\mathbb{R}^m$;*
2. *$T$ is one-to-one iff the columns of $A$ are linearly independent.*

**PROOF.**

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**PROOF.**

We prove earlier that columns of $A$ span $\mathbb{R}^m$ iff for each $b \in \mathbb{R}^m$ the equation $Ax = b$ is consistent i.e. $T(x) = b$ has solution.
This is true iff $T$ is $\mathbb{R}^n$ onto $\mathbb{R}^m$. □

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**PROOF.**

$T(x)$ is one-to-one iff $Ax = 0$ has only trivial solution. This is true only if columns of $A$ are linearly independent. □