Inverse of a matrix

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Row-Column Rule

THEOREM

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In other words if $(AB)_{i,j}$ denotes (i, j)-entry of AB and A is $m \times n$ matrix, then

$$a_{i,1}b_{1,j} + \cdots + a_{i,n}b_{n,j}$$

.

Matrix Multiplication

EXAMPLE

Let us compute AB where $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$.

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$$Ab_1 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \cdot 2 + 1 \cdot 3 \\ 4 \cdot 1 + 1 \cdot (-5) \end{bmatrix},$$

$$Ab_2 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \cdot 2 + (-2) \cdot 3 \\ 3 \cdot 1 + (-2) \cdot (-5) \end{bmatrix},$$

$$Ab_3 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \cdot 2 + 3 \cdot 3 \\ 6 \cdot 1 + 3 \cdot (-5) \end{bmatrix}.$$

Matrix Multiplication

THEOREM

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + BC$
3. $(B + C)A = BA + CA$
4. $r(AB) = (rA)B = A(rB)$
5. $I_m A = A = AI_n$

Powers of Matrix

DEFINITION

If A is $n \times n$ matrix, then A^k denotes the product of k copies of A.

The Transpose of Matrix

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If A is $m \times n$ matrix, then the **transpose** of A, denoted by A^T is a matrix whose columns are formed from the corresponding rows of A.

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Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

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If A is $m \times n$ matrix, then the **transpose** of A, denoted by A^T is a matrix whose columns are formed from the corresponding rows of A.

THEOREM

Let A and B denote matrices whose size are appropriate for the following operations.

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. *for any scalar r,* $(rA)^T = r(A^T)$

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NOTE

Note that there is only one such C since if there is also $B \neq C$ such that $BA = I_n$, then $B = BI_n = BAC = I_nC = C$, contradiction.

We call such C as A^{-1}.

The Inverse of a Matrix

THEOREM

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. *If* $ad - bc \neq 0$*, then* A *is invertible and*

$$A = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -cc & a \end{bmatrix}$$

and if $ad - bc = 0$*, then* A *is not invertible.*

THEOREM

If A *is invertible* $n \times n$ *matrix, then for each* b *in* $\mathbb{R}^n$*, the equation* $Ax = b$ *has the unique solution* $x = A^{-1}b$*.*

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PROOF.

Take any $b \in \mathbb{R}^n$. Solution of the equation $Ax = b$ exists since we can substitute $A^{-1}b$ instead of x.

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Take any $b \in \mathbb{R}^n$. Solution of the equation $Ax = b$ exists since we can substitute $A^{-1}b$ instead of x.
In order to prove that solution u is unique let us note that $A^{-1}Au = A^{-1}b$, hence $u = A^{-1}b$. □

The Inverse of a Matrix

THEOREM

1. *If A is invertible matrix, then A^{-1} is invertible and $(A^{-1})^{-1} = A$.*
2. *If A and B are invertible $n \times n$ matrices, then so is AB and $(AB)^{-1} = B^{-1}A^{-1}$.*
3. *If A is invertible matrix, then A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.*

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PROOF.

The definition of invertible matrix states that $AC = CA = I_n$ but we can consider this equality as a statement that A is inverse of C.

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PROOF.

Note that $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I_n$.

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PROOF.

Note that $A^T(A^{-1})^T = (A^{-1}A)^T = I_n$. □

An Elementary Matrix

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An **elementary matrix** is one that is obtained by preforming single elementary row operation on an identity matrix.

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REMARK

If an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be written as EA, where the $m \times n$ matrix E is created by performing the same row operation on I_m.

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And in this case any sequence of operations that transform A to I_n also transforms I_n to A^{-1}.

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Let us assume that A is invertible. In this case $Ax = b$ has solution for any b, hence A has pivot position on each row. As a result, since A is square, pivot position are on the main diagonal of A.
Hence the reduced echelon form of A is I.

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Hence the reduced echelon form of A is I.
Now suppose conversely $A \sim I_n$. Hence there is a sequence E_1, …, E_p of elementary matrices such that $E_1 E_2 \dots E_p A = I_n$.
But it means, that $A = (E_1 \dots E_p)^{-1}$ □