Vector Spaces and Subspaces

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Elementary matrices and Invertibility

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Hence the reduced echelon form of A is I.
Now suppose conversely $A \sim I_n$. Hence there is a sequence E_1, ..., E_p of elementary matrices such that $E_1 E_2 \dots E_p A = I_n$.
But it means, that $A = (E_1 \dots E_p)^{-1}$. □

An Algorithm for Finding A^{-1}

ALGORITHM

Row reduce $\begin{bmatrix} A & I \end{bmatrix}$. If A is row equivalent to I, then $\begin{bmatrix} A & I \end{bmatrix}$ is row equivalent to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$. Otherwise A does not have an inverse.

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$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1.5 & -2 & 0.5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -4.5 & 7 & -1.5 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 1.5 & -2 & 0.5 \end{bmatrix}$$

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10. *There is an $n \times n$ matrix D such that $AD = I$.*

Invertible Linear Transformations

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A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is **invertible** iff there exists a function S such that

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Suppose T is invertible. Since $T(S(x)) = x$ for all $x \in \mathbb{R}^n$, T is onto $\mathbb{R}^n$.

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Thus A is invertible.
Conversely implication is easy to see. □

Vector Spaces and Subspaces

DEFINITION

A vector space is a non empty set V of objects called vectors for which are defined two operations, called addition and multiplication by scalars subject to the ten axioms.

- The sum $u + v$ of two vectors $u, v \in V$ belongs to V;
- $u + v = v + u$;
- $(u + v) + w = u + (v + w)$;
- there is a zero vector 0 in V such that $u + 0 = 0 + u = u$;
- For each vector $u \in V$ there is a vector $-u$ such that $u + (-u) = 0$;
- the scalar multiple of u by c, denoted by cu, is in V;
- $c(u + v) = cu + cv$;
- $(c + d)u = cu + du$;
- $c(du) = (cd)u$;
- $1u = u$.

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$$\{y_k\}_{k \in \mathbb{Z}} = (\ldots, y_{-1}, y_0, y_1, \ldots).$$

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Let $\mathbb{S}$ be a set of double infinity sequences of real numbers.

$$\{y_k\}_{k\in\mathbb{Z}} = (\dots, y_{-1}, y_0, y_1, \dots).$$

If $\{z_k\}_{k\in\mathbb{Z}}$ is another element of $\mathbb{S}$, then
$\{y_k\}_{k\in\mathbb{Z}} + \{z_k\}_{k\in\mathbb{Z}} = \{y_k + z_k\}_{k\in\mathbb{Z}}$ and $c\{y_k\}_{k\in\mathbb{Z}} = \{cy_k\}_{k\in\mathbb{Z}}$.

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For $n \geq 1$, the set $\mathbb{P}_n$ of polynomials of degree at most n consists of all polynomials of the form:

$$p(t) = a_0 + a_1 t + \cdots + a_n t^n$$

where the coefficients a_0, ..., a_n and the variable t are real numbers. The zero polynomial is a polynomial with all a_i equal to zero.

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If $q(t) = b_0 + b_1 t + \cdots + b_n t^n$ be another polynomial, then the sum $p + q$ is defined by

$$(p + q)(t) = p(t) + q(t) = (a_0 + b_0) + (a_1 + b_1)t + \cdots + (a_n + b_n)t^n.$$

The scalar multiple cp is the polynomial defined by:

$$(cp)(t) = cp(t) = ca_0 + ca_1 t + \cdots + ca_n t^n.$$

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Let V be a set of real-valued functions defined on some set D. Functions are added in a usual way: $(f + g)(x) = f(x) + g(x)$ and $(cf)(x) = cf(x)$.

Subspaces

DEFINITION

The subspace of a vector space V is a set U such that

- the zero vector is in U;
- U is closed under addition i.e. for any $u, v \in U$ $u + v \in U$;
- U is closed under multiplication i.e. for any $u \in U$ and $c \in \mathbb{R}$ $cu \in U$.

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Let $\mathbb{P}$ be a set of all polynomials with real coefficients with operations defined as for functions.
$\mathbb{P}$ is a subspace of the space of real-valued functions defined on $\mathbb{R}$.
Also, $\mathbb{P}_n$ is a subspace of $\mathbb{P}$ for all $n \geq 1$.

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DEFINITION

The set $\text{Span}\{v_1, \ldots, v_p\}$ is a set of all linear combinations of vectors v_1, ..., v_p.

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- $\text{Span}\{v_1, v_2\}$ is closed under multiplication since $c(s_1 v_1 + s_2 v_2) = (cs_1)v_1 + (cs_2)v_2$.

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If v_1, ..., v_p are vectors from a vector space V, then $\text{Span}\{v_1, \ldots, v_p\}$ is a subspace of V.