Null spaces and column spaces

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Null Space

DEFINITION

The **null space** of an $m \times n$ matrix A, written as Nul A, is the solution set of the equation $Ax = 0$. In set notation,

$$\text{Nul } A = \{x : x \in \mathbb{R}^n \text{ and } Ax = 0\}.$$

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Let us determine if $u \in \text{Nul } A$.
To test it we need to check if $Au = 0$:

$$Au = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 - 9 + 4 \\ -25 + 27 - 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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   $A(u + v) = Au + Av = 0 + 0 = 0$.

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3. Let $Au = 0$ and $c \in \mathbb{R}$. $A(cu) = cAu = c0 = 0$.

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Hence H is a subspace of $\mathbb{R}^4$.

An Explicit Description of Nul A

EXAMPLE

Let us find a spanning set for the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

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Let us reduce it to echelon form.

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{aligned} x_1 - 2x_2 \qquad - \quad x_4 + 3x_5 &= 0 \\ x_3 + 2x_4 - 2x_5 &= 0 \end{aligned}$$

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$$x = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 10 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

The Column Space of a Matrix

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The **column space** of a matrix $A = [a_1 \dots a_n]$, written as $\text{Col } A$ is a set of all linear combinations of columns of A i.e.

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Note that $\begin{bmatrix} 6a - b \\ a + b \\ -7a \end{bmatrix} = a \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. Hence, $A = \begin{bmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{bmatrix}$.

Linear Transformation

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A **linear transformation** from a vector space V into a vector space W is a rule that assigns to each vector $x \in V$ a unique vector $T(x) \in W$, such that

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Note that if $T(x) = Ax$ is a matrix transformation, then kernel is the same as null space and range is a column space.

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Note that the kernel of this transformation is a set of sequences $\{a_i\}_{i \in \mathbb{N}}$ such that $a_{i+2} = a_{i+1} + a_i$. For example, Fibonacci numbers belongs to the kernel.

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D is a linear transformation. The kernel of D is a set of all constant functions. The range of D is W.