Linear Independent Sets, Bases

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- The set $\{\sin, \cos\}$ is linearly independent since sin and cos are not multiples of one another. Since if $\sin = c\cos$, then for all x $\sin(x) = c\cos(x)$ but it is true for 0 only if $c = 0$.

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- the subspace spanned by $\mathfrak{B}$ coincides with H.

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- The set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^3$.

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- The set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$ spans $\mathbb{R}^3$ but linearly dependent.

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Note that if A is an invertible $n \times n$ matrix, then the equation $Ax = b$ has a solution for every $b \in \mathbb{R}^n$. Hence $\text{Col } A = \mathbb{R}^n$ and columns of A span $\mathbb{R}^n$.

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Note that if A is an invertible $n \times n$ matrix, then columns of A are linearly independent. □

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The set of vectors $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, ..., $e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ from $\mathbb{R}^n$ is called a **standard basis** for $\mathbb{R}^n$.

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The Spanning Set Theorem

EXAMPLE

Let $v_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 6 \\ 16 \\ -5 \end{bmatrix}$.

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Note that $v_3 = 5v_1 + 3v_2$. Hence $\text{Span}\{v_1, v_2, v_3\} = \text{Span}\{v_1, v_2\}$.
Also v_1 and v_2 are linearly independent vectors. So $\{v_1, v_2\}$ is a basis of $\text{Span}\{v_1, v_2, v_3\}$.

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- *If* $H \neq \{0\}$*, then some subset of* S *is a basis of* H*.*

Bases for Nul A and Col A

EXAMPLE

Let us fin a basis of column space of a matrix

$$B = \begin{bmatrix} b_1 & b_2 & \dots & b_5 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Each pivot column of is a linear combination of nonpivot. For example $b_2 = 4b_1$ and $b_4 = 2b_1 - b_3$.

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By previous theorem we may discard b_2 and b_4. Hence

$$\text{Col } A = \text{Span}\{b_1, b_3, b_5\} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$