Dimension of a vector space and Rank

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Bases for Nul A and Col A

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EXAMPLE

Let us find a spanning set for the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -3 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

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Let us reduce it to echelon form.

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & -2 \end{bmatrix}, \quad \begin{aligned} x_1 \qquad\quad - x_4 + 3x_5 &= 0 \\ x_2 \qquad\qquad\qquad &= 0 \\ x_3 + 2x_4 - 2x_5 &= 0 \end{aligned}$$

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$$x = \begin{bmatrix} x_4 - 3x_5 \\ 0 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

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$$\text{Nul } A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Coordinates

THEOREM

Let $\mathfrak{B} = \{v_1, \ldots, v_n\}$ *be a basis for V. If* $c_1 v_1 + \ldots c_n v_n = d_1 v_1 + \cdots + d_n v_n$*, then* $c_i = d_i$ *for all i.*

PROOF.

Let for some i $c_i \neq d_i$. Then not all $c_i - d_i = 0$ but $(c_1 - d_1)v_1 + \ldots (c_n - d_n)v_n = 0$, hence $\mathfrak{B}$ is a linearly dependent set, contradiction. □

Coordinates

DEFINITION

Let $\mathfrak{B} = \{v_1, \ldots, v_n\}$ be a basis for V.
For any vector $v \in V$ there are unique c_1, ..., c_n such that $v = c_1 v_1 + \cdots + c_n v_n$. We denote a vector $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ by $[v]_{\mathfrak{B}}$.

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Note that for any basis $\mathfrak{B}$ the transformation $[v]_{\mathfrak{B}}$ is a linear transformation with a kernel $\{0\}$.

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EXAMPLE

Let $\mathfrak{B} = \{b_1, b_2\}$ be a basis for $\mathbb{R}^2$ where $b_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $b_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

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$$x = (-2) \cdot b_1 + 3 \cdot b_2 = (-2) \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}.$$

Coordinates

REMARK

Note that the entries of a vector $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ are the coordinates of x relative to the standard basis.

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Let $b_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $b_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $x = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, and $\mathfrak{B} = \{b_1, b_2\}$.

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$[x]_{\mathfrak{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

Change of Coordinates Matrices

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Note that $P_{\mathfrak{B}}$ is invertible and $P_{\mathfrak{B}}^{-1} x = [x]_{\mathfrak{B}}$.

The Coordinate Mapping

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Let $\mathfrak{B} = \{b_1, \ldots, b_n\}$ be a basis for a vector space V. Then the coordinate mapping $x \mapsto [x]_{\mathfrak{B}}$ is one-to-one linear transformation from V onto $\mathbb{R}^n$.

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Note that $[u_1]_{\mathfrak{B}}$, ..., $[u_p]_{\mathfrak{B}}$ are linearly dependent. Hence there are d_1, ..., d_p such that $d_1[u_1]_{\mathfrak{B}} + \cdots + d_p[u_p]_{\mathfrak{B}} = 0$.

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But kernel of this transformation is 0. Hence $d_1 u_1 + \cdots + d_p u_p = 0$. □