Dimension of a vector space and Rank

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The Dimension of a Vector Space

THEOREM

If a vector space V has a basis $\mathfrak{B} = \{v_1, \ldots, v_n\}$, then any set in V containing more than n vectors is linearly dependent.

The Dimension of a Vector Space

PROOF.

Let us fix some set $S = \{u_1, \dots, u_p\}$ with $p > n$.
In this proof we denote $[u]_{\mathfrak{B}}$ by $T(u)$ ($T\colon V \to \mathbb{R}^n$).

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Hence, there are d_1, ..., d_p such that $d_1 T(u_1) + \dots + d_p T(u_p) = 0$.

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Hence, there are d_1, ..., d_p such that $d_1\, T(u_1) + \cdots + d_p T(u_p) = 0$.
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But kernel of this transformation is 0. Hence $d_1 u_1 + \cdots + d_p u_p = 0$. $\square$

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DEFINITION

If a vector space V is spanned by a finite set, then V is a **finite-dimensional** space and the **dimension** of V, written as $\dim V$, is a number of elements in a basis of V.
We define dimension of $\{0\}$ space as 0.

Subspaces of a Finite-Dimensional Space

THEOREM

Let H be a subspace of a finite-dimensional space V.

- *Any linearly independent set in H can be expanded to a basis of V and*
- $\dim H \leq \dim V$.

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But S is a subset of V, hence $|S| \leq n$. Hence eventually S will span H and be a basis of H. $\square$

The Row Space

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Note that it the same as $\text{Col } A^T$.

The Row Space

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- *If two matrices A and B are row equivalent, then* $\text{Row } A = \text{Row } B$.
- *If a matrix A is in echelon form, then nonzero rows form basis of* $\text{Row } A$.

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$\text{rank } A = \dim \text{Row } A$. □

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It means that $\dim \text{Nul } A = 2$. In other words $\text{rank } A = 40$. But $\text{Col } A \subseteq \mathbb{R}^{40}$. Thus for every b the system $Ax = b$ has a solution.