Change of Basis

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Invertible Matrix Theorem

THEOREM

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5. $\text{Nul } A = \{0\}$;
6. $\dim \text{Nul } A = 0$.

Change of Basis

EXAMPLE

Let $\mathfrak{B} = \{b_1, b_2\}$ be a basis for $\mathbb{R}^2$ where $b_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $b_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

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Suppose $x \in \mathbb{R}^2$ such that $[x]_{\mathfrak{B}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

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$$x = (-2) \cdot b_1 + 3 \cdot b_2 = (-2) \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}.$$

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Let $P_{\mathfrak{B}} = \begin{bmatrix} b_1 & b_2 & \dots & b_n \end{bmatrix}$.

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Let $P_{\mathfrak{B}} = \begin{bmatrix} b_1 & b_2 & \dots & b_n \end{bmatrix}$. Then $x = P_{\mathfrak{B}}[x]_{\mathfrak{B}}$.

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Consider two bases $\mathfrak{B} = \{b_1, b_2\}$ and $\mathfrak{C} = \{c_1, c_2\}$ for a vector space V.

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Let us find $[x]_{\mathfrak{C}}$.

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Since $[x]_{\mathfrak{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ we know that $x = 3b_1 + b_2$.

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$[x]_{\mathfrak{C}} = 3[b_1]_{\mathfrak{C}} + [b_2]_{\mathfrak{C}}$.

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As a result $[x]_{\mathfrak{C}} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$.

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The columns of $P_{\mathfrak{C} \leftarrow \mathfrak{B}}$ *are* $\mathfrak{C}$*-coordinates of the vectors in the basis* $\mathfrak{B}$*.*

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That is, $P_{\mathfrak{C} \leftarrow \mathfrak{B}} = \begin{bmatrix} [b_1]_{\mathfrak{C}} & \dots & [b_n]_{\mathfrak{C}} \end{bmatrix}$.
The matrix $P_{\mathfrak{C} \leftarrow \mathfrak{B}}$ is called ***change-of-coordinates matrix from $\mathfrak{B}$ to $\mathfrak{C}$****.*

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REMARK

Note that

$$[x]_{\mathfrak{C}} = P_{\mathfrak{C} \leftarrow \mathfrak{B}} [x]_{\mathfrak{B}}$$

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As a result $P_{\mathfrak{B}\leftarrow\mathfrak{C}}$ is invertible and $P_{\mathfrak{B}\leftarrow\mathfrak{C}}^{-1} = P_{\mathfrak{C}\leftarrow\mathfrak{B}}$

Change of Basis in $\mathbb{R}^n$

EXAMPLE

Let $b_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$, $b_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$, $c_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, and $c_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$.

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Let $[b_1]_{\mathfrak{C}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $[b_2]_{\mathfrak{C}} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

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Let $[b_1]_{\mathfrak{C}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $[b_2]_{\mathfrak{C}} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$
By definition

$$\begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b_1, \quad \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = b_2$$

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In order to solve this system simultaneously, let us consider

$$\left[\begin{array}{cc|cc} c_1 & c_2 & b_1 & b_2 \end{array} \right] = \left[\begin{array}{cc|cc} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{array} \right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{array} \right].$$

Change of Basis in $\mathbb{R}^n$

Recall that for each $x \in \mathbb{R}^n$,

$$P_{\mathfrak{B}}[x]_{\mathfrak{B}} = x, \qquad P_{\mathfrak{C}}[x]_{\mathfrak{C}} = x, \qquad \text{and } P_{\mathfrak{C}}^{-1}x = [x]_{\mathfrak{C}}.$$

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As a result, $P_{\mathfrak{C}\leftarrow\mathfrak{B}} = P_{\mathfrak{C}}^{-1}P_{\mathfrak{B}}$.

Introduction to the Determinant

Consider an invertible matrix $A = [a_{i,j}]$ such that $a_{1,1} \neq 0$.

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$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \sim \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{1,1}a_{2,1} & a_{1,1}a_{2,2} & a_{1,1}a_{2,3} \\ a_{1,1}a_{3,1} & a_{1,1}a_{3,2} & a_{1,1}a_{3,3} \end{bmatrix} \sim$$

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ 0 & a_{1,1}a_{2,2} - a_{2,1}a_{1,2} & a_{1,1}a_{2,3} - a_{2,1}a_{1,3} \\ 0 & a_{1,1}a_{3,2} - a_{3,1}a_{1,2} & a_{1,1}a_{3,3} - a_{3,1}a_{1,3} \end{bmatrix} \sim$$

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where $\Delta = a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} -$

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