Determinants

Authors:
Alexander Knop

Institute:
UC San Diego

Introduction to Determinants

Consider a matrix $A = [a_{i,j}]$ of size 2×2 and its determinant $\det A = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$.

Introduction to Determinants

Consider a matrix $A = [a_{i,j}]$ of size 2×2 and its determinant $\det A = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$.
We know that A is invertible iff

Introduction to Determinants

Consider a matrix $A = [a_{i,j}]$ of size 2×2 and its determinant $\det A = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$.
We know that A is invertible iff $\det A \neq 0$.

Introduction to Determinants

Consider an invertible matrix $A = [a_{i,j}]$ such that $a_{1,1} \neq 0$.

Introduction to Determinants

Consider an invertible matrix $A = [a_{i,j}]$ such that $a_{1,1} \neq 0$.

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \sim \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{1,1}a_{2,1} & a_{1,1}a_{2,2} & a_{1,1}a_{2,3} \\ a_{1,1}a_{3,1} & a_{1,1}a_{3,2} & a_{1,1}a_{3,3} \end{bmatrix} \sim$$

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ 0 & a_{1,1}a_{2,2} - a_{2,1}a_{1,2} & a_{1,1}a_{2,3} - a_{2,1}a_{1,3} \\ 0 & a_{1,1}a_{3,2} - a_{3,1}a_{1,2} & a_{1,1}a_{3,3} - a_{3,1}a_{1,3} \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ 0 & a_{1,1}a_{2,2} - a_{2,1}a_{1,2} & a_{1,1}a_{2,3} - a_{2,1}a_{1,3} \\ 0 & 0 & a_{1,1}\Delta \end{bmatrix}$$

Introduction to Determinants

Consider an invertible matrix $A = [a_{i,j}]$ such that $a_{1,1} \neq 0$.

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \sim \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{1,1}a_{2,1} & a_{1,1}a_{2,2} & a_{1,1}a_{2,3} \\ a_{1,1}a_{3,1} & a_{1,1}a_{3,2} & a_{1,1}a_{3,3} \end{bmatrix} \sim$$

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ 0 & a_{1,1}a_{2,2} - a_{2,1}a_{1,2} & a_{1,1}a_{2,3} - a_{2,1}a_{1,3} \\ 0 & a_{1,1}a_{3,2} - a_{3,1}a_{1,2} & a_{1,1}a_{3,3} - a_{3,1}a_{1,3} \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ 0 & a_{1,1}a_{2,2} - a_{2,1}a_{1,2} & a_{1,1}a_{2,3} - a_{2,1}a_{1,3} \\ 0 & 0 & a_{1,1}\Delta \end{bmatrix}$$

where $\Delta = a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} -$

$a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3} - a_{1,3}a_{2,2}a_{3,1}.$

Introduction to Determinants

$$\Delta = a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} - a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3} - a_{1,3}a_{2,2}a_{3,1}.$$

Recall that the determinant of a 2×2 matrix $A = [a_{i,j}]$ is $\det A = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$.

Introduction to Determinants

$$\Delta = a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} - a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3} - a_{1,3}a_{2,2}a_{3,1}.$$

Recall that the determinant of a 2×2 matrix $A = [a_{i,j}]$ is $\det A = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$.

Hence $\Delta = a_{1,1} \det \begin{bmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{bmatrix} - a_{1,2} \det \begin{bmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{bmatrix} + a_{1.3} \det \begin{bmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{bmatrix} = a_{1,1} \det A_{1,1} - a_{1,2} \det A_{1,2} + a_{1,3}A_{1,3}$

The Definition of the Determinant

DEFINITION

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{i,j}]$ is the following

$$a_{1,1} \det A_{1,1} - a_{1,2} A_{1,2} + \cdots + (-1)^{n+1} a_{1,n} \det A_{1,n}$$

The Definition of Determinant

EXAMPLE

Let us compute the determinant of $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$.

The Definition of Determinant

EXAMPLE

Let us compute the determinant of $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$.

$\det A = 1 \det \begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix} - 5 \det \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} + 0 \det \begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix} =$
$1(0-2) - 5(0-0) + 0(-4-0) = -2$

Another Definition of the Determinant

DEFINITION

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{i,j}]$ is the following

$$a_{1,1} \det A_{1,1} - a_{1,2} \det A_{1,2} + \cdots + (-1)^{n+1} a_{1,n} \det A_{1,n}$$

Another Definition of the Determinant

DEFINITION

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{i,j}]$ is the following

$$a_{1,1} \det A_{1,1} - a_{1,2} \det A_{1,2} + \cdots + (-1)^{n+1} a_{1,n} \det A_{1,n}$$

DEFINITION

Let $C_{i,j} = (-1)^{i+j} \det A_{i,j}$.

Another Definition of the Determinant

DEFINITION

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{i,j}]$ is the following

$$a_{1,1} \det A_{1,1} - a_{1,2} \det A_{1,2} + \cdots + (-1)^{n+1} a_{1,n} \det A_{1,n}$$

DEFINITION

Let $C_{i,j} = (-1)^{i+j} \det A_{i,j}$.
For $n \geq 2$, the determinant of an $n \times n$ matrix $A = [a_{i,j}]$ is the following

$$a_{1,1} C_{1,1} + a_{1,2} C_{1,2} + \cdots + a_{1,n} C_{1,n}.$$

A Cofactor Expansion

THEOREM

For $n \geq 2$ and $1 \leq i \leq n$, the determinant of an $n \times n$ matrix $A = [a_{i,j}]$ is equal to

$$a_{i,1}C_{i,1} + a_{i,2}C_{i,2} + \cdots + a_{i,n}C_{i,n}$$

and in the same time to

$$a_{1,i}C_{1,i} + a_{2,i}C_{2,i} + \cdots + a_{n,i}C_{n,i}.$$

A Cofactor Expansion

EXAMPLE

Let us compute $\det \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$

A Cofactor Expansion

EXAMPLE

Let us compute $\det \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix} = a_{3,1}C_{3,1} + a_{3,2}C_{3,2} + a_{3,3}C_{3,3} =$

A Cofactor Expansion

EXAMPLE

Let us compute $\det \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix} = a_{3,1} C_{3,1} + a_{3,2} C_{3,2} + a_{3,3} C_{3,3} =$

$- 2 \det \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} =$

A Cofactor Expansion

EXAMPLE

Let us compute $\det \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix} = a_{3,1}C_{3,1} + a_{3,2}C_{3,2} + a_{3,3}C_{3,3} =$

$- 2 \det \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = - 2(-1 - 0) = -2$

A Cofactor Expansion

EXAMPLE

Let us compute $\det \begin{bmatrix} 3 & -7 & 89 & -6 & \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$

A Cofactor Expansion

EXAMPLE

Let us compute $\det \begin{bmatrix} 3 & -7 & 89 & -6 & \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix} =$

$3 \det \begin{bmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{bmatrix}$

A Cofactor Expansion

EXAMPLE

Let us compute $\det \begin{bmatrix} 3 & -7 & 89 & -6 & \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix} =$

$3 \det \begin{bmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{bmatrix} = 6 \det \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$

A Cofactor Expansion

EXAMPLE

Let us compute $\det \begin{bmatrix} 3 & -7 & 89 & -6 & \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix} =$

$$3 \det \begin{bmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{bmatrix} = 6 \det \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix} = -12$$

A Cofactor Expansion

THEOREM

If an $n \times n$ matrix A is a triangular matrix, then the determinant of A equal to the product of diagonal elements of A.

Row Operations

THEOREM

Let A be a square matrix.

- *If a multiple of one row of A is added to another row to produce a matrix B, then* $\det B = \det A$.

Row Operations

THEOREM

Let A be a square matrix.

- *If a multiple of one row of A is added to another row to produce a matrix B, then* $\det B = \det A$.
- *If two rows of A interchanged to produce B, then* $\det B = -\det A$.

Row Operations

THEOREM

Let A be a square matrix.

- *If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.*
- *If two rows of A interchanged to produce B, then $\det B = -\det A$.*
- *If one row of A is multiplied by k to produce B, then $\det B = k \det A$.*

Row Operations

EXAMPLE

Let us compute $\det \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix} =$

Row Operations

EXAMPLE

Let us compute $\det \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix} = \det \begin{bmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ -1 & 7 & 0 \end{bmatrix}$

Row Operations

EXAMPLE

Let us compute $\det \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix} = \det \begin{bmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ -1 & 7 & 0 \end{bmatrix} =$

$\det \begin{bmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{bmatrix}$

Row Operations

EXAMPLE

Let us compute $\det \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix} = \det \begin{bmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ -1 & 7 & 0 \end{bmatrix} =$

$\det \begin{bmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{bmatrix} = -\det \begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{bmatrix}$

Row Operations

EXAMPLE

Let us compute $\det \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix} = \det \begin{bmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ -1 & 7 & 0 \end{bmatrix} =$
$\det \begin{bmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{bmatrix} = -\det \begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{bmatrix} = 15.$

Row Operations

EXAMPLE

Let us compute $\det \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & -6 \end{bmatrix} =$

Row Operations

EXAMPLE

Let us compute $\det \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & -6 \end{bmatrix} = 2 \det \begin{bmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & -6 \end{bmatrix}$

Row Operations

EXAMPLE

Let us compute $\det \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & -6 \end{bmatrix} = 2 \det \begin{bmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & -6 \end{bmatrix} =$

$2 \det \begin{bmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{bmatrix} =$

Row Operations

EXAMPLE

Let us compute $\det \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & -6 \end{bmatrix} = 2\det \begin{bmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & -6 \end{bmatrix} =$

$$2\det \begin{bmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{bmatrix} =$$

$$2\det \begin{bmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & -2 \\ 0 & 0 & -3 & 2 \end{bmatrix} 2\det \begin{bmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Row Operations

EXAMPLE

Let us compute $\det\begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & -6 \end{bmatrix} = 2\det\begin{bmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & -6 \end{bmatrix} =$

$2\det\begin{bmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{bmatrix} =$

$2\det\begin{bmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & -2 \\ 0 & 0 & -3 & 2 \end{bmatrix} 2\det\begin{bmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 2 \cdot 3 \cdot (-6) \cdot 1 =$

$-36.$

The Determinant and the Echelon Form

THEOREM

Let A be a square matrix and suppose that A has been transformed into echelon form U using r row interchanges.

The Determinant and the Echelon Form

THEOREM

Let A be a square matrix and suppose that A has been transformed into echelon form U using r row interchanges.
Then $\det A = (-1)^r \det U$.

The Determinant and the Echelon Form

THEOREM

Let A be a square matrix and suppose that A has been transformed into echelon form U using r row interchanges.
Then $\det A = (-1)^r \det U$.

$$\det A = \begin{cases} (-1)^r(\textit{product of pivots in } U) & \textit{when } A \textit{ is invertible} \\ 0 & \textit{when } A \textit{ is not invertible} \end{cases}$$

The Determinant and the Echelon Form

THEOREM

Let A be a square matrix and suppose that A has been transformed into echelon form U using r row interchanges.
Then $\det A = (-1)^r \det U$.

$$\det A = \begin{cases} (-1)^r(\textit{product of pivots in } U) & \textit{when } A \textit{ is invertible} \\ 0 & \textit{when } A \textit{ is not invertible} \end{cases}$$

THEOREM

A square matrix A is invertible iff $\det A \neq 0$.