Determinants and Volume

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Hence the cofactor expansion of the first row of A is equal to the cofactor expansion of the first column of A^T. $\square$

The Multiplicative Property

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Let A and B be $n \times n$ matrices. Then $\det AB = \det A \det B$.

EXAMPLE

Let $A = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$.

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$\det AB = 25 \cdot 13 - 20 \cdot 14 = 45$ and $(\det A) \cdot (\det B) = 9 \cdot 5 = 45$.

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- *If one row of A is multiplied by k to produce B, then $\det B = k \det A$.*

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If A is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then $\det EA = (\det E)(\det A)$ *where*

$$\det E = \begin{cases} 1 & \text{if } E \text{ is a row replacement} \\ -1 & \text{if } E \text{ is an interchange} \\ r & \text{if } E \text{ is a scale by } r \end{cases}$$

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$\det EA = a_{i,1} \det B_{i,1} - a_{i,2} \det B_{i,2} + \cdots + (-1)^n a_{i,n} \det B_{i,n}$

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Let A be an invertible $n \times n$ matrix. For any $b \in \mathbb{R}^n$, the unique solution x of $Ax = b$ has entries defined by $x_i = \frac{\det A_i(b)}{\det A}$.

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Since $\det A = 3s^2 - 12 = 3(s+2)(s-2)$ the system has unique solution when $s \neq \pm 2$.

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For such s the solution is

$$x_1 = \frac{\det A_1(b)}{\det A} = \frac{4s+2}{3(s+2)(s-2)},$$

$$x_2 = \frac{\det A_2(b)}{\det A} = \frac{3s+24}{3(s+2)(s-2)} = \frac{s+8}{(s+2)(s-2)}.$$