Eigenvectors and Eigenvalues

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Let $Ax = b$. Note that

$$A \cdot I_i(x) = A \begin{bmatrix} e_1 & \dots & x & \dots & e_n \end{bmatrix} = \begin{bmatrix} Ae_1 & \dots & Ax & \dots & Ae_n \end{bmatrix} = \begin{bmatrix} a_1 & \dots & b & \dots & a_n \end{bmatrix} = A_i(b).$$

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$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{1,1} & C_{2,1} & \dots & C_{n,1} \\ C_{1,2} & C_{2,2} & \dots & C_{n,2} \\ \vdots & \vdots & & \vdots \\ C_{1,n} & C_{2,n} & \dots & C_{n,n} \end{bmatrix}.$$

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DEFINITION

We call the matrix on the right side an **adjugate** of A and denote it $\operatorname{adj} A$.

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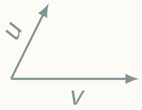

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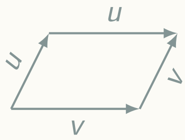

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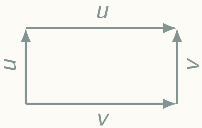

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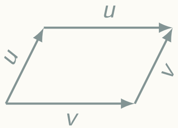

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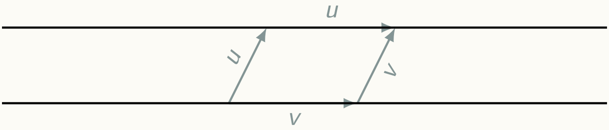

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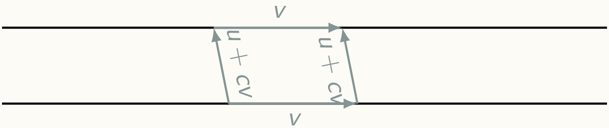

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THEOREM

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ *be a linear transformation and* A *be its stadard matrix. Then for any parallepiped* S

$$\{\mathit{area\ of\ } T(S)\} = |\det A| \cdot \{\mathit{area\ of\ } S\}.$$

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PROOF.

Let $A = \begin{bmatrix} a_1 & a_2 \end{bmatrix}$ and

$$S = \{s_1 b_1 + s_2 b_2 \ : \ 0 \leq s_1, s_2 \leq 1\}.$$

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The image of S under the transformation T consists of the points

$$T(s_1 b_1 + s_2 b_2) = s_1 T(b_1) + s_2 T(b_2) = s_1 A b_1 + s_2 A b_2,$$

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$$\{\text{area of } T(S)\} = |\det AB| = |\det A| \cdot |\det B| = |\det A| \cdot \{\text{area of } S\}.$$

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Hence, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a nontrivial solution.

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The set of all x such that $(A - \lambda I)x = 0$ is called an **eigenspace** of A corresponding to λ.

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any eigenvector corresponding to 2 has the following form

$$x \begin{bmatrix} 1/2 \\ 10 \end{bmatrix} + y \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

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Hence, if $\lambda \neq a_{1,1}$ and $\lambda \neq a_{2,1}$, then this system does not have free variables and does not have nontrivial solutions. □

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If v_1, ..., v_r are eigenvectors that corresponds to distinct eigenvalues λ_1, ..., λ_r of $n \times n$ matrix A, then the set $\{v_1, \ldots, v_r\}$ is linearly independent.

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$$Av_p = \lambda_p v_p = c_1 A v_1 + \dots c_r A v_r = c_1 \lambda_1 v_1 + \dots c_{p-1} \lambda_{p-1} v_{p-1}.$$

But since $v_p = c_1 v_1 + \dots c_{p-1} v_{p-1}$ we may conclude that $0 = c_1(\lambda_1 - \lambda_p) v_1 \dots c_{p-1}(\lambda_{p-1} - \lambda_p) v_{p-1}$. Hence v_1, ..., v_{p-1} are linearly dependant, contradisction. □