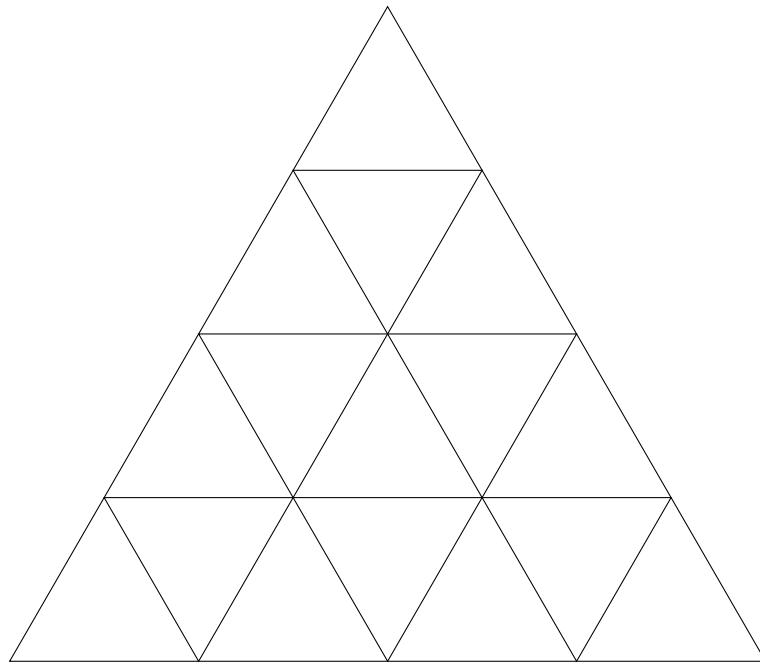


Name: _____

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1. (15 points) We are given 17 points inside a regular triangle T of side length 1. Prove that two of these points have distance $\leq 1/4$ apart.

Solution: In order to do it let us consider the following partition of the triangle.



Note that there are 16 small triangles inside, hence by pigeonhole principle there are at least two points in the same small triangle. Additionally, distance between any two points inside the regular triangle with side $1/4$ is at most $1/4$.

2. (20 points) A certain academic term contains 40 days of class. A teacher needs to give 5 exams on these days, but any two of these exams must be ≥ 3 class days apart (for example, if there is an exam on Day 12, then there can be no exam on Day 13 or Day 14, but an exam on Day 15 is alright). How many ways can the teacher do this?

Solution: Note that the calendar of the semester should be like this
 X_1 Exam₁ Day Day X_2 Exam₂ Day Day ... Exam₅ X_6 where X_i is some number of days.
In other words we need to “put” 40 – 13 days into 6 “boxes” X_1, \dots, X_6 . Hence, the answer is
 $\binom{40-13+6-1}{6-1}$.

3. (20 points) Give a simple closed form expression for the sum

$$\sum_{\substack{a+b+c=7 \\ a,b,c \geq 0}} \binom{7}{a,b,c}$$

Solution: During the lectures we proved that

$$\sum_{\substack{a+b+c=7 \\ a,b,c \geq 0}} \binom{7}{a,b,c} x^a y^b z^c = (x + y + z)^7.$$

Hence,

$$\sum_{\substack{a+b+c=7 \\ a,b,c \geq 0}} \binom{7}{a,b,c} = (1 + 1 + 1)^7.$$

4. (a) (5 points) Define the Stirling number of the second kind $S(n, k)$.

Solution: $S(n, k)$ is a number of partitions of $[n]$ into k nonempty subsets.

- (b) (20 points) Let n and k be positive integers. Give a formula for the number of surjections $f : [n] \rightarrow [k]$ in terms of $S(n, k)$.

Solution: The number of surjective functions is equal to $k! \cdot S(n, k)$. Since for each partition $\{P_1, \dots, P_k\}$ and a permutation $\pi : [k] \rightarrow [k]$ corresponds a surjective function $f : [n] \rightarrow [k]$ by the following rule: $f(i) = j$ iff $i \in P_{\pi(j)}$. Note that this correspondence is a bijection, hence we proved that the number of surjective functions is equal to $k! \cdot S(n, k)$.