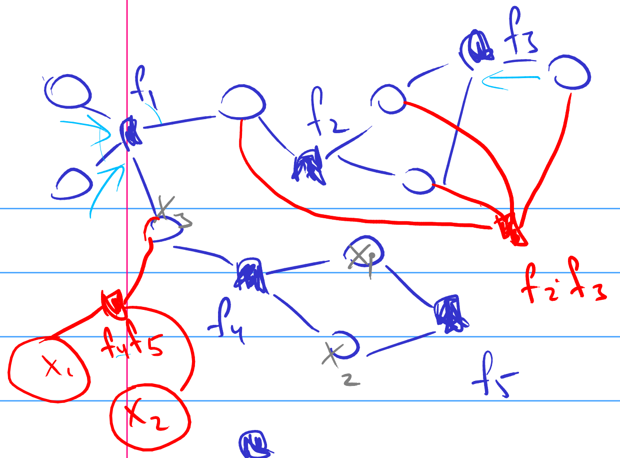


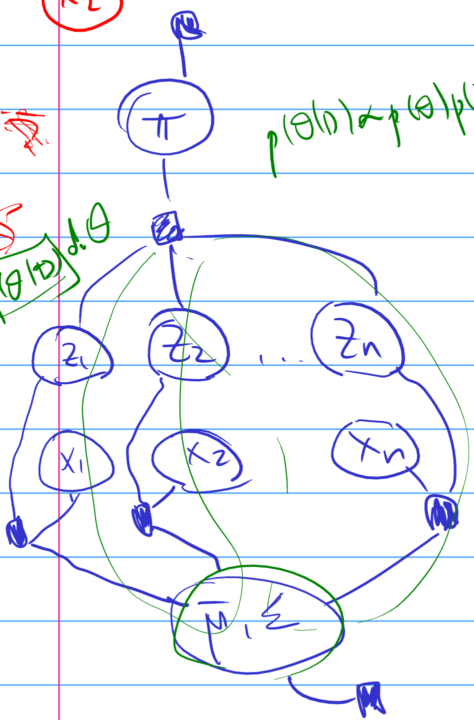
$F(X) = f_1 f_2 f_3 f_4 f_5$

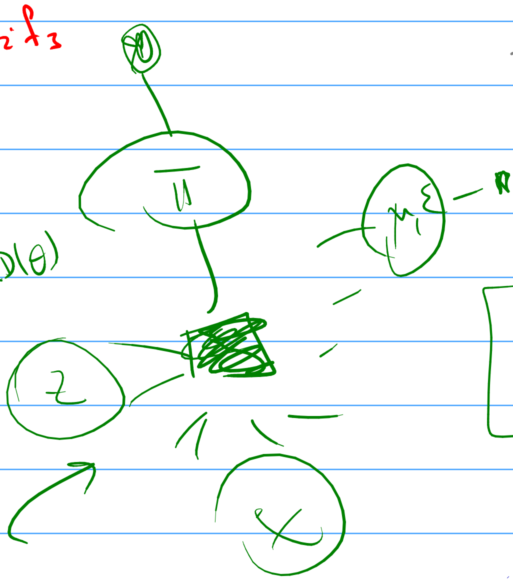
$\underline{f_4(x_1,x_2,x_3)\, f_5(x_1,x_2)}$
$\quad\quad =$
$f_{45}(x_1,x_2,x_3)$

$\int p(x|\theta)\, p(\theta|D)\, d\theta$

$\frac{\hat{\sum} p(x|\theta^{?})}{R}$

$p(\theta|D) \propto p(\theta)\, p(D|\theta)$

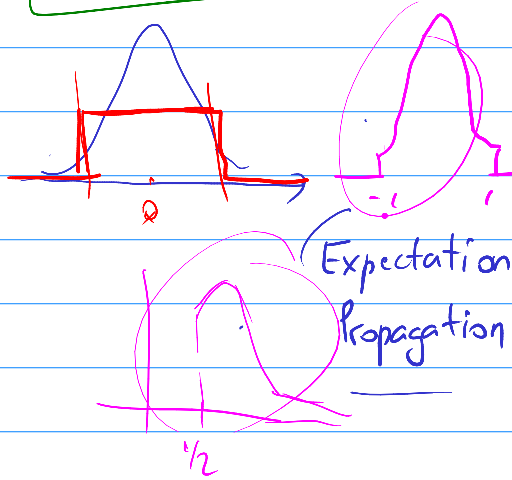




$$\boxed{\mu_{f \to x} = \int f(x,\bar{y}) \prod \mu_{y_j \to f}\, d\bar{y}}$$

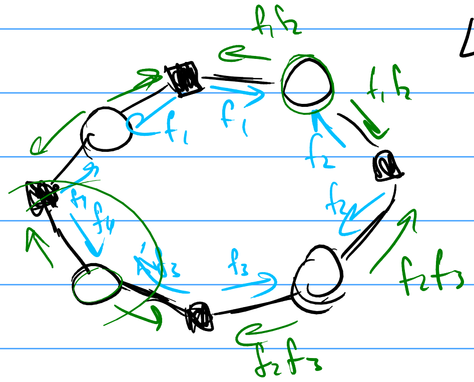
$\bar{\pi} \sim p(\bar{\mu}|D)$

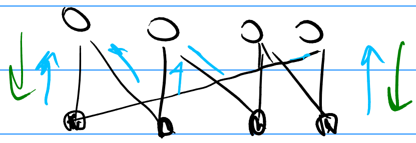
$\mu^* = \frac{1}{R} \sum \mu^{(2)}$



Expectation Propagation

Loopy Belief Propagation



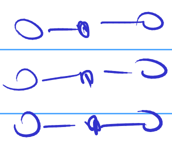


Variational approximations

$p(x_1 \ldots x_n)$

$\approx$

$q(x_1 \ldots x_n) = \prod q_i(x_i)$



Sampling

$\boxed{p(\theta|D)}$
$=$
$\bar{x}$ — $\square$ — $p(\bar{x})$

$\bar{x}$ — $\square$ — $p^*(\bar{x}) \propto p(\bar{x})$
$=$
$p(\theta)\, p(D|\theta)$

Unif[0,1] → $\square$ → $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_R$
rand()
$\sim p(\bar{x})$

$$p(\theta|D) = \frac{p(\theta)p(D|\theta)}{p(D)} \qquad 1)\ p(\theta|D) \xrightarrow[\theta]{} \max$$

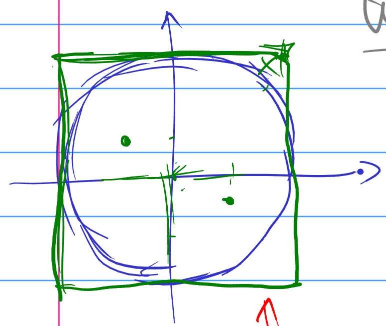
$$2)\ p(x|D) = \int p(x|\theta)\,p(\theta|D)\,d\theta = \mathbb{E}_{p(\theta|D)}\left[p(x|\theta)\right]$$

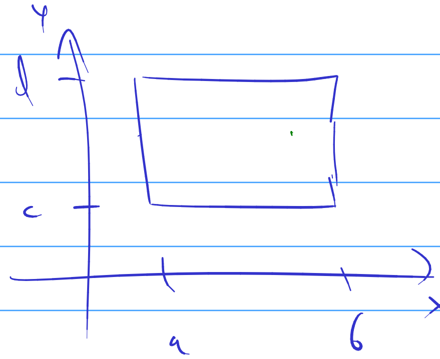
$$p(\theta)\prod_n p(d_n|\theta) = p^*(\bar{x}) \propto p(\bar{x}) \rightsquigarrow \mathbb{E}_{p(\bar{x})}[f(\bar{x})] \approx \frac{1}{R}\sum_{r=1}^{R} f(\bar{x}^{(r)}),\quad \bar{x}^{(r)} \sim p(\bar{x})$$

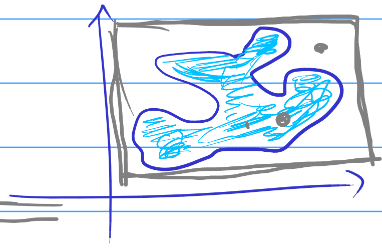
1) Известные распределения

$x \sim \mathrm{Unif}([0,1]) \xrightarrow{b+(b-a)x} x \sim \mathrm{Unif}([a,b])$?

$\mathrm{Unif}(D)$

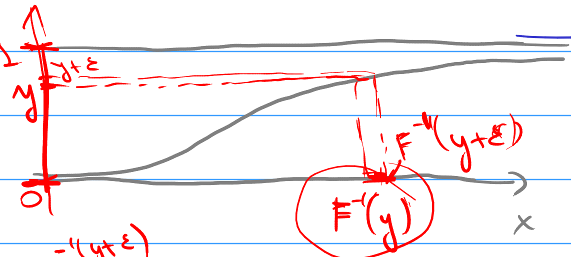






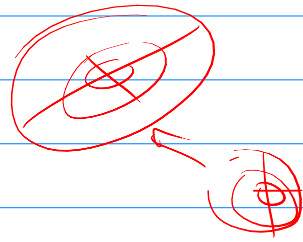
$x \in \mathbb{R}$ $\quad p(x)$

$F(a) = \int_{-\infty}^{a} p(x)\,dx$

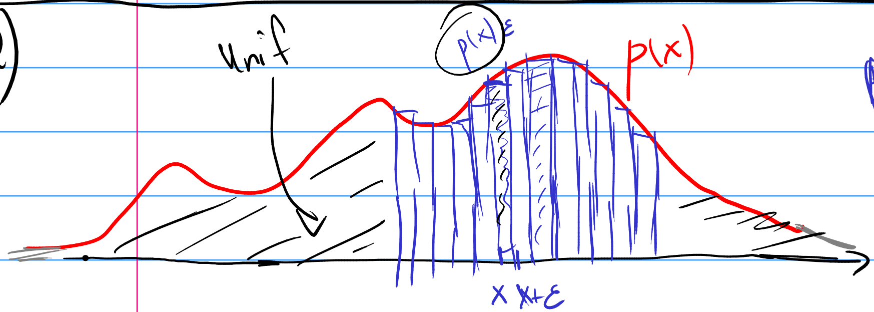


$p(y \in [y, y+\varepsilon]) = \varepsilon$

$$\int_{F^{-1}(y)}^{F^{-1}(y+\varepsilon)} p(x) = p(x \in [F^{-1}(y), F^{-1}(y+\varepsilon)])$$



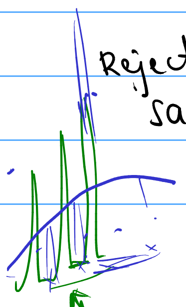
2)

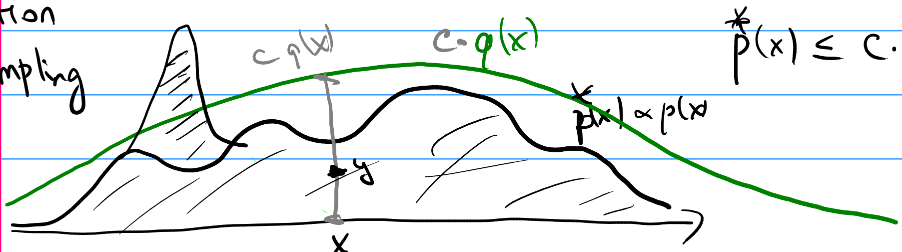


$Pr(x \in [a,b]) = \int_a^b p(x)\,dx = F(b) - F(a)$

$Pr([x, x+\varepsilon]) = \int_x^{x+\varepsilon} p(x)\,dx \approx p(x)\cdot\varepsilon$

Rejection sampling





$p^*(x) \le c\cdot q(x)$

- sample $x \sim q(x)$
- sample $y \sim [0, c\,q(x)]$
- if $y < p^*(x)$ accept
  else reject

$p(\theta)\prod_n p(d_n|\theta) \Leftarrow$ curse of dimensionality

Importance sampling

$$\mathbb{E}_{p(\bar{x})}[f(\bar{x})] = \int f(\bar{x})p(\bar{x})d\bar{x} = \int f(\bar{x})\frac{p(\bar{x})}{q(\bar{x})}\cdot q(\bar{x})d\bar{x} =$$

$\bar{x} \sim q(\bar{x})$

$$= \mathbb{E}_{q(\bar{x})}\left[f(\bar{x})\cdot\frac{p(\bar{x})}{q(\bar{x})}\right] \approx \frac{1}{R}\sum_r f(\bar{x}^{(r)})\cdot\frac{p(\bar{x}^{(r)})}{q(\bar{x}^{(r)})}$$

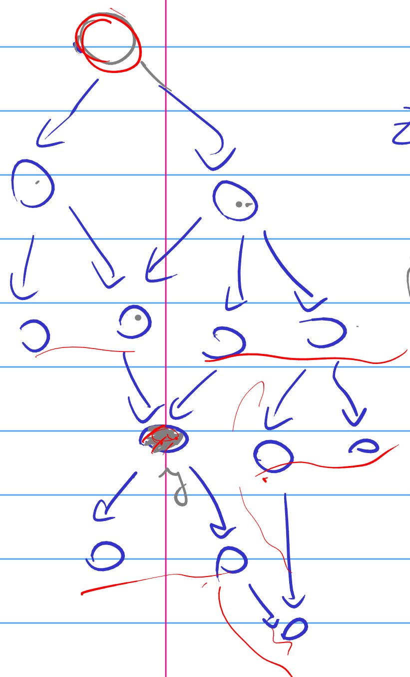
importance weights

$$p(\bar{x}) = \frac{1}{Z_p}\cdot p^*(\bar{x})$$

$$Z_p = \int p^*(\bar{x})d\bar{x} \approx \frac{1}{R}\sum_r \frac{p^*(\bar{x}^{(r)})}{q(\bar{x}^{(r)})}$$

$$Z_q \approx \frac{1}{R}\sum_r \frac{q^*(\bar{x}^{(r)})}{q(x^{(r)})}$$

$$\frac{p}{q} = \frac{p^*}{q^*}\cdot\frac{Z_q}{Z_p} \approx \frac{\sum q^*}{\sum p^*}$$



$$p(\bar{x}) = \prod_i p(x_i \mid par(x_i))$$

$$p(\bar{x} \mid y) \qquad p(y \mid par(y))$$

---

3) MCMC Markov Chain Monte-Carlo

$$X_1 \to X_2 \to X_3 \to \dots \qquad p(x_t \mid x_{t-1}) = p(x_t \mid x_{t-1}, \dots, x_1)$$

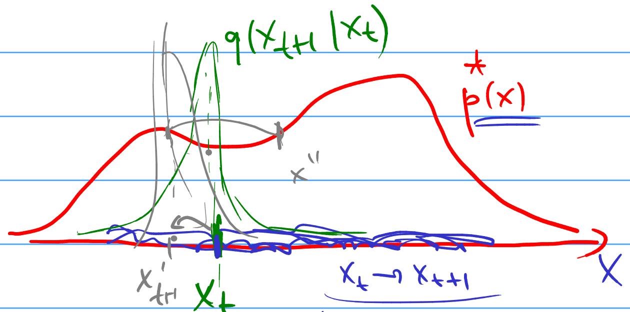
$$\pi(x) = \int p(x \mid y)\pi(y)\,dy$$

Условие баланса

если

$\forall x, y \quad \boxed{p(x)\,q(y \mid x) = p(y)\,q(x \mid y)}$

то $p(x)$ – стац. распр. для цепи q



$q(x_t \mid x_{t-1})$

$p(x)$

$$p(x) \stackrel{?}{=} \int q(x \mid y)\,p(y)\,dy = \int p(x)\,q(y \mid x)\,dy$$

Metropolis-Hastings algorithm:

– repeat:

– sample $x' \sim q(x' \mid x_t)$

– $a = \dfrac{p^*(x')}{p^*(x)}\cdot\dfrac{q(x \mid x')}{q(x' \mid x)}$

– if $a \geq 1$ then $x_{t+1} = x'$

else $x_{t+1} = \begin{cases} x' & \text{с вер. } a \\ x_t & \text{с вер. } 1-a \end{cases}$

x, x'

$a(x,x') \geq 1$

$$p(x) \cdot p(x'|x) = p(x')\, p(x|x')$$

$$p(x'|x) = \cancel{q(x'|x)}$$

$$p(x|x') = \cancel{q(x|x')} \cdot \frac{p^*(x)}{p^*(x')} \cdot \frac{\cancel{q(x'|x)}}{\cancel{q(x|x')}}$$

Gibbs sampling

$p(x_1 \ldots x_n)$

$p(x_i|\bar{x}_{-i}) = p(x_i | x_1 \ldots x_{i-1}, x_{i+1}, \ldots, x_n)$

- repeat:
  - for $i = 1..n$

    $x_i \sim p(x_i|\bar{x}_{-i})$

$x_1^{(t)}, x_2^{(t)}, \ldots, x_n^{(t)}$

$x_1^{(t+1)}, x_2^{(t)}, \ldots, x_n^{(t)}$

$x_1^{(t+1)}, x_2^{(t+1)}, \ldots$

$p(Z \,|\, X)$

$p(z_n|\bar{z}_{-n}, X)$

$$q: (\underbrace{\bar{x}_{1..i-1}^{(t+1)}, x_i^{(t)}, \bar{x}_{i+1..n}^{(t)}}_{\bar{x}_{-i}}) \longmapsto (\bar{x}_{\ldots}^{(t+1)}, x_i^{(t+1)}, \bar{x}_{\ldots}^{(t)})$$

$$q(x_i', \bar{x}_{-i} \,|\, x_i, \bar{x}_{-i}) = p(x_i'|\bar{x}_{-i})$$

$$a = \frac{p(x_i', \bar{x}_{-i})}{p(x_i, \bar{x}_{-i})} \cdot \frac{q(x_i, \bar{x}_{-i} \,|\, x_i', \bar{x}_{-i})}{q(x_i', \bar{x}_{-i} \,|\, x_i, \bar{x}_{-i})} =$$

$$= \frac{\cancel{p(x_i'|\bar{x}_{-i})}\,\cancel{p(\bar{x}_{-i})}}{\cancel{p(x_i|\bar{x}_{-i})}\,\cancel{p(\bar{x}_{-i})}} \cdot \frac{\cancel{p(x_i|\bar{x}_{-i})}}{\cancel{p(x_i'|\bar{x}_{-i})}} = 1$$