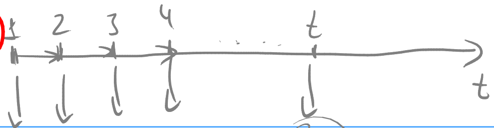
SIR - модель - susceptible - infected - recovered

$x_i^{(t)} \in \{S, I, R\}$

$X = \{x_1, \dots, x_N\}$

① $S \to I \to R$

②



$y^{(1)} \quad \dots \quad y^{(t)}$ - число зарег. больных в момент t

$t: \; S^{(t)}, I^{(t)}, R^{(t)}$

$|\{i \in 1..N \mid x_i^{(t)} = S\}|$

$\theta = \{\beta, \mu, p, \pi\}$

1) $p(x_i^{(1)} = I) = \pi_i, \quad p(x_i^{(1)} = S) = 1 - \pi_i$

2) $p(x_i^{(t)} \in y^{(t)} \mid x_i^{(t)} = I) = p$

$p(y^{(t)} \mid I^{(t)}, p) = \text{Binomial}(y^{(t)} \mid I^{(t)}, p)$

3) $p(x_i^{(t+1)} = R \mid x_i^{(t)} = I) = \mu$

4) β ③ $= p$(заразиться от одного контакта)

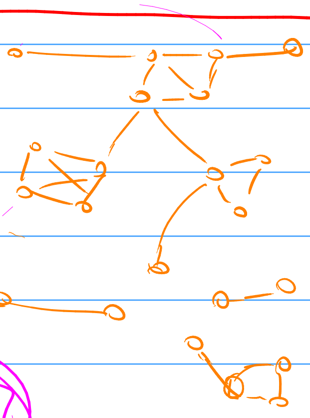
$p(x_i^{(t+1)} = S \mid x_i^{(t)} = S) = (1-\beta)^{I^{(t)}}$

$p(x_i^{(t+1)} = I \mid x_i^{(t)} = S) = 1 - (1-\beta)^{I^{(t)}}$

① SEIR, SEIRS

② Непрерывное время

③ Модель заражения



$$p(x_i^{(t+1)} \mid x_i^{(t)}) = \begin{array}{c|ccc} & & & \\ S^{(t)} & (1-\beta)^{I^{(t)}} & 1-(1-\beta)^{I^{(t)}} & 0 \\ I & 0 & 1-\mu & \mu \\ R & 0 & 0 & 1 \\ & S & I & R^{(t+1)} \end{array}$$

R_0 — $\beta \cdot E[\#\text{конт.}]$

$X = \{x_i^{(t)} \mid i, t\}$

$y = \{y^{(t)} \mid t\}$

$$p(X, y \mid \theta) = p(X^{(1)} \mid \pi)\, p(X^{(2)} \mid X^{(1)}, \beta, \mu) \dots p(X^{(T)} \mid X^{(T-1)}, \beta, \mu) \cdot$$

$$\cdot\, p(y^{(1)} \mid X^{(1)}, p) \dots p(y^{(T)} \mid X^{(T)}, p) =$$

$$= \left(\prod_i \pi^{[x_i^{(1)} = I]} (1-\pi)^{[x_i^{(1)} = S]} \right) \cdot \prod_{t=1}^{T-1} \prod_{i=1}^{N} p(x_i^{(t+1)} \mid I^{(t)}, \beta, \mu) \cdot$$

$$\cdot \left(\prod_{t=1}^{T} \binom{I^{(t)}}{y^{(t)}} p^{y^{(t)}} (1-p)^{I^{(t)} - y^{(t)}} \right)$$

$$\underset{\text{max}}{p(\theta \mid y)} \propto p(\theta)\, \underset{\text{max}}{p(y \mid \theta)} = p(\theta) \cdot \int p(X, y \mid \theta)\, dX =$$

$$= p(\theta) \int \underline{p(X \mid y, \theta)}\, p(y \mid \theta)\, dX$$

$$p(\theta \mid y) = \mathbb{E}_{p(X \mid \theta)} [p(\theta)\, p(y \mid X, \theta)] \approx \frac{1}{R} \sum_{r=1}^{R} p(\underline{\theta})\, p(y \mid \underline{X_r}, \underline{\theta})$$

$$X_r \sim p(X \mid \theta, y)$$

EM fix $\theta_0 \xrightarrow{\text{Gibbs}} X_0^{(r)}$

MAP

$\theta_1 \xrightarrow{\text{Gibbs}} X_1^{(r)}$

$\theta^* \dots$ MAP

$X \sim p(X \mid y, \theta)$

$\;\; \shortparallel$

$\{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_N\}$

$\;\; \shortparallel$

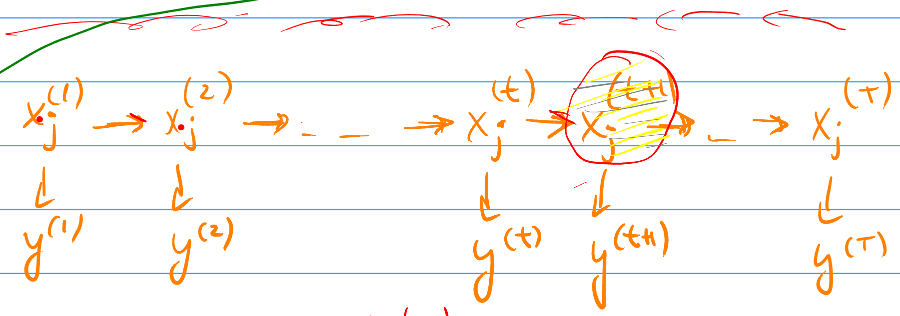
$(x_1^{(1)}, x_1^{(2)}, \dots, x_1^{(T)})$

Gibbs sampling:

$j=1$ ... $j=N$: $\boxed{\bar{x}_j \sim p(\bar{x}_j \mid \boxed{X_{-j}}, y, \theta)}$

Hidden Markov models:

Stochastic Viterbi algorithm



$p(\bar{x}_j \mid X_{-j}, \bar{y}, \theta)$

$$p(x_j^{(t)} \mid x_j^{(t+1)}, X_{-j}, \bar{x}_j^{(t+2)..(T)}, \bar{y}, \theta)$$

$$= p(x_j^{(t)} \mid x_j^{(t+1)}, X_{-j}, \bar{y}^{(1)..(t)}, \theta)$$

$$q^{(t)}_{j,s',s} = p(x_j^{(t)} = s, x_j^{(t-1)} = s' \mid \bar{y}^{(1)..(t)}, X_{-j}, \theta)$$

$x_j^{(1)}$

$p(x_j^{(1)} = s \mid \dots) \propto \pi_i(s) \cdot p(y^{(1)} \mid x_j^{(1)} = s)$

$$q^{(t)}_{j,s',s} = p(x_j^{(t)} = s, x_j^{(t-1)} = s' \mid \bar{y}, X_{-j}, \theta) = p(x_j^{(t)} = s \mid x_j^{(t-1)} = s', \cancel{\bar{y}}, X_{-j}, \theta) \cdot$$

$$\cdot\, p(y^{(t)} \mid \underline{x_j^{(t)} = s}, \cancel{x_j^{(t-1)} = s'}, \cancel{y^{(1..t-1)}}, X_{-j}, \theta) \cdot$$

$$\cdot\, p(\underline{x_j^{(t-1)} = s'} \mid y^{(1)..(t-1)}, \underline{X_{-j}}, \theta) =$$

$$\shortparallel$$

$$\underbrace{\sum_{s''} p(x_j^{(t-1)} = s', x_j^{(t-2)} = s'' \mid \bar{y}, X_{-j}, \theta)}_{q}$$

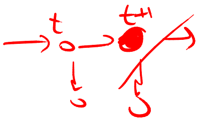
$$= p(x_j^{(t)} = s \mid x_j^{(t-1)} = s', X_{-j}, \theta) \cdot \mathrm{Binom}(y^{(t)} \mid I_{-j}^{(t)} + [s=1], \theta) \cdot$$

$$\cdot \left(\sum_{s''} q^{(t-1)}_{j,s'',s'} \right)$$

1) Послед. Схема $Q_j^{(t)} = \begin{array}{c|c} & S \\ \hline \begin{array}{c} S \\ I \\ R \\ \uparrow \\ s' \end{array} & \begin{pmatrix} & \vert & \\ - & q^{(t)}_{j,s',s} & - \\ & \vert & \end{pmatrix} \end{array}$

$$p(\bar{x}_j \mid \dots) = p(x_j^{(T)} \mid \dots)\, p(x_j^{(T-1)} \mid x_j^{(T)}, \dots)\, p(x_j^{(T-2)} \mid \underline{x_j^{(T-1)}}, \cancel{x_j^{(T)}}, \dots) \dots p(x_j^{(1)} \mid x_j^{(2)}, \cancel{x^{(T)}} \dots)$$

2) Сэмпл. $x_j^{(T)}, x_j^{(T-1)}, \dots, x_j^{(1)}$



$$p(x_j^{(T)} = s | y, X_{-j}, \theta) = \sum_{s'} p(x_j^{(T)} = s, x_j^{(T-1)} = s' | ..) = \sum_{s'} q_{j,s',s}^{(T)}$$

$$p(x_j^{(t)} = s | x_j^{(t+1)} = s', y, X_{-j}, \theta) = p(x_j^{(t)} = s | x_j^{(t+1)} = s', y^{(1)..(t)}, X_{-j}, \theta) = q_{j,s,s'}^{(t+1)}$$

Алгоритм:

1) Init X так, чтобы Y могли получиться

Gibbs

2) Повторяем:
  - выб. j
  - строим $Q_j^{(t)} = (q_{j,s,s'}^{(t)})_{s,s' \in \{S,I,R\}}$
  - сэмпл. $\bar{x}_j \sim p(\bar{x}_j | X_{-j}, y, \theta)$ при пом. Q
  - замен. $\bar{x}_j$ на новый

3) Обновл. пар-ры модели при тек. X $\quad \theta = \{p, \mu, \beta, \pi\}$

$$p(\theta | y, X) \propto \underbrace{p(\theta)} p(y, X | \theta)$$
$$'' \; p(p)p(\mu)p(\beta)p(\pi)$$

$$p(p) = Beta(p | a_p, b_p)$$
$$\dots \mu, \beta, \boxed{\pi} \dots$$

$$\log p(\theta | y, X) = const + (a_p - 1)\log p + (b_p - 1)\log(1-p) + (a_\mu - 1)\log\mu + (b_\mu - 1)\log(1-\mu)$$
$$+ (a_\beta - 1)\log\beta + (b_\beta - 1)\log(1-\beta) + (a_\pi - 1)\log\pi + (b_\pi - 1)\log(1-\pi)$$

$$+ \sum_{i=1}^{N} \left[[x_i^{(1)} = I] \log\pi + [x_i^{(1)} = S] \log(1-\pi) \right]$$

$$a_p' = a_p + \sum_{t=1}^{T} y^{(t)}$$
$$b_p' = b_p + \sum_t (I^{(t)} - y^{(t)})$$

$$+ \sum_{t=1}^{T} \left(y^{(t)} \log p + (I^{(t)} - y^{(t)}) \log(1-p) \right)$$

$$+ \sum_{i=1}^{N} \sum_{t=1}^{T-1} \left([x_i^{(t)} = I, x_i^{(t+1)} = R] \log\mu + [x_i^{(t)} = I, x_i^{(t+1)} = I] \log(1-\mu) \right)$$

$$+ \sum_{t=1}^{T-1} \sum_{i: x_i^{(t)} = S}' \left(\underbrace{P_i^{(t)}} \log\beta + N_i^{(t)} \log(1-\beta) \right)$$

$P_i^{(t)}$ = # контактов, от к-рых зар.

$N_i^{(t)}$ = # конт., от к-рых не заразился

$$P_i^{(t)} + N_i^{(t)} = \underline{I^{(t)}}$$

Presence-only data

- если $x_i^{(t+1)} = S$, то $P_i^{(t)} = 0$, $N_i^{(t)} = I^{(t)}$
- если $\underline{x_i^{(t+1)} = I}$, то $\underline{P_i^{(t)} \geq 1}$, $N_i^{(t)} = I^{(t)} - P_i^{(t)}$

$$E\left[P_i^{(t)} | x_i^{(t+1)} = I\right] = I^{(t)} \cdot p(\text{зараж.} | \geq 1 \text{ зараж. было}) = I^{(t)} \cdot \frac{\beta}{1 - (1-\beta)^{I^{(t)}}}$$

$$P_i^{(t)} = \begin{cases} 0, & x_i^{(t+1)} = S \\ I^{(t)} \dfrac{\beta}{1-(1-\beta)^{I^{(t)}}}, & x_i^{(t+1)} = I \end{cases}, \quad N_i^{(t)} = I^{(t)} - P_i^{(t)}$$

$$a_\beta' = a_\beta + \sum_{t=1}^{T-1} \sum_{i : x_i^{(t)} = S} p_i^{(t)} \qquad b_\beta' = b_\beta + \_\_ \, N_i^{(t)}$$