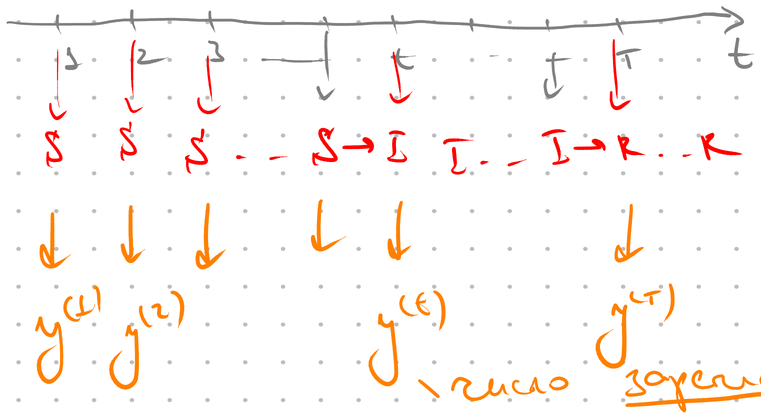
SIR susceptible – infected – recovered

$\overline{x}_j = \left(x_j^{(1)}, \dots, x_j^{(T)}\right)$

$X = \{x_1, \dots, x_N\}$

$\mathcal{S} = \{S, I, R\}$

$t: \quad x_j^{(t)} \in \{S, I, R\}$



$S^{(t)} = |\{j \in 1,..,N \mid x_j^{(t)} = S\}|$

$I^{(t)} = \text{— " —} \quad I$

$R^{(t)} = \text{— " —} \quad R$

$S^{(t)} + I^{(t)} + R^{(t)} = N$

$y^{(t)}$ – число <u>зарегистр.</u> больных во время t

---

1) $p(x_j^{(1)} = I) = \underline{\pi_j}, \quad p(x_j^{(1)} = S) = 1 - \underline{\pi_j} \qquad \overline{\pi} = (\pi_1, \dots, \pi_N)$

2) $p(x_j^{(t)} \in y^{(t)} \mid x_j^{(t)} = I) = \textcircled{\rho}$

$$p(y^{(t)} \mid I^{(t)}, \rho) = \boxed{\text{Binomial}\left(\underline{y^{(t)} \mid I^{(t)}, \rho}\right)}$$

3) $p(x_j^{(t+1)} = R \mid x_j^{(t)} = I) = \textcircled{\mu} \qquad \underline{p(x_j^{(t+1)} = I \mid x_j^{(t)} = I) = 1 - \mu}$

4) $\textcircled{\beta}$ = p(заразиться от одного контакта)

$\overline{\theta} = \{\overline{\pi}, \rho, \mu, \beta\}$

$$p(x_j^{(t+1)} = S \mid x_j^{(t)} = S) = (1-\beta)^{I^{(t)}}$$

$$p(x_j^{(t+1)} = I \mid x_j^{(t)} = S) = 1 - (1-\beta)^{I^{(t)}}$$

$$p(x_j^{(t+1)} \mid x_j^{(t)}) = \begin{array}{c|ccc} S & (1-\beta)^{\textcircled{I^{(t)}}} & 1-(1-\beta)^{\textcircled{I^{(t)}}} & 0 \\ I & 0 & 1-\mu & \mu \\ R & 0 & 0 & 1 \\ \hline & S & I & R \end{array}$$

$$p(X, Y \mid \overline{\theta}) = p(X^{(1)} \mid \pi)\, p(y^{(1)} \mid X^{(1)}, \rho) \cdot p(X^{(2)} \mid X^{(1)}, \mu, \beta) \dots$$

$$\dots p(X^{(T)} \mid X^{(T-1)}, \mu, \beta)\, p(y^{(T)} \mid X^{(T)}, \rho) =$$

$X^{(t)} = \{x_1^{(t)}, \dots, x_N^{(t)}\}$

S, I, R

$$p(y^{(t)} \mid x^{(t)}, \mu, \beta)$$

$$= \left(\prod_{i=1}^{N} \pi^{[x_i^{(1)}=I]} (1-\pi)^{[x_i^{(1)}=S]} \right) \times \left(\prod_{t=1}^{T} \left[\binom{I^{(t)}}{y^{(t)}} \rho^{y^{(t)}} (1-\rho)^{I^{(t)}-y^{(t)}} \right] \right)$$

$$\times \left(\prod_{t=1}^{T-1} \prod_{i=1}^{N} p\left(x_i^{(t+1)} \mid x_i^{(t)}, I^{(t)}, \beta, \mu\right) \right)$$

$$\underline{p(\bar{\theta} \mid y)} = p(\bar{\pi}, \rho, \beta, \mu \mid y^{(1)}, \ldots, y^{(T)}) \xrightarrow[\bar{\pi}, \rho, \beta, \mu]{} \max$$

$$\propto \underline{p(\theta)}\, p(y \mid \theta) = \underline{p(\theta)} \int p(x, y \mid \theta)\, dx$$

EM-algorithm

$$L(\theta, \theta_k) = \mathbb{E}_{p(x \mid \theta_k, y)} \left[\underline{p(x, y \mid \theta)\, p(\theta)} \right] \xrightarrow[\theta]{} \max$$

$$\theta_{k+1} = \arg\max [L]$$

Monte Carlo EM:

$$L(\theta, \theta_k) \approx \frac{1}{R} \sum_{r=1}^{R} p(X_r, y \mid \theta)\, p(\theta), \text{ ge}$$

$$X_r \sim p(X \mid \underline{\underline{\theta_k}}, y)$$

$$X = \{\bar{x}_1, \ldots, \bar{x}_N\}$$

$$" \{x_1^{(1)}, \ldots, x_1^{(T)}\}$$

$$\begin{tikzcd}
\theta_0 \arrow[r, "\text{sample}"] & X_{0,r} \arrow[dl, "\text{MAP}"'] \\
\theta_1 \arrow[r, "\text{sample}"] & X_{1,r} \arrow[dl, "\text{MAP}"'] \\
\theta_2 \arrow[r] & \ldots
\end{tikzcd}$$

Sampling

$$X \sim p(X \mid \theta, y)$$

$$\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_N\}$$

$$\boxed{\bar{x}_j \sim p(\bar{x}_j \mid \theta, y, \underline{\underline{X_{-j}}})}$$

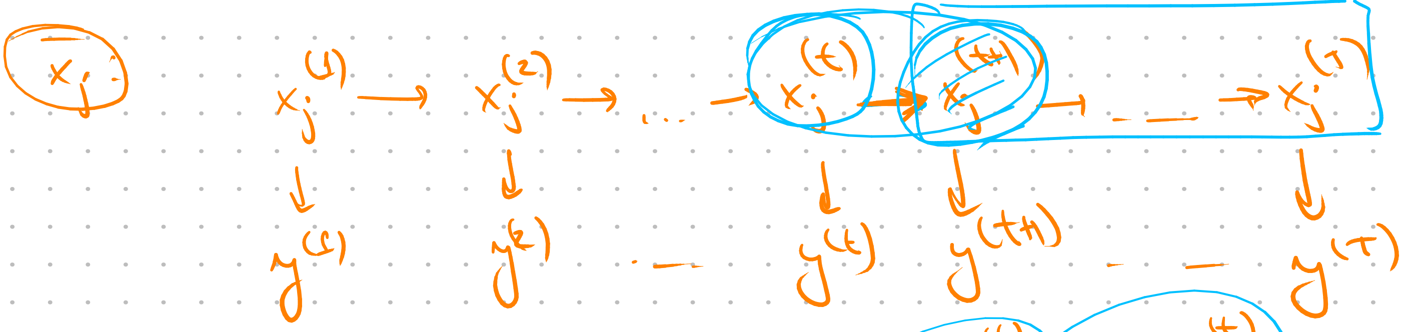
Gibbs sampling

$j=1$ $\quad \bar{x}_1 \sim \ldots$

$j=2$ $\quad \bar{x}_2 \sim \ldots$

$\vdots$

$j=N$ $\quad \bar{x}_N \sim \ldots$



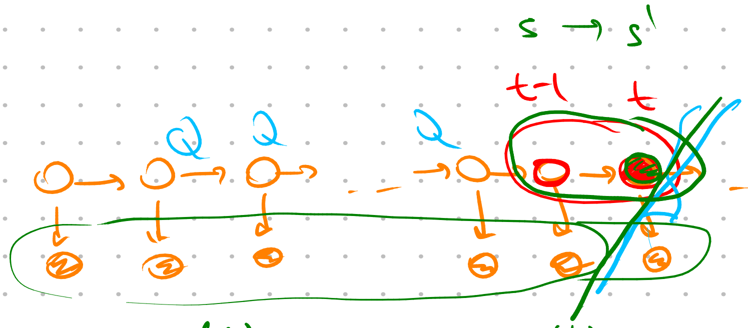
$$p(x_j^{(t+1)} | x_j^{(t)}, \theta, X_{-j}) = \begin{pmatrix} (1-\beta)^{I^{(t)}} & 1-(1-\beta)^{I^{(t)}} & 0 \\ 0 & 1-\mu & \mu \\ 0 & 0 & 1 \end{pmatrix}$$

$$\bar{x}_j \sim p(\bar{x}_j | \theta, X_{-j}, y)$$

Stochastic Viterbi algorithm

$$p(\bar{x}_j | X_{-j}, y, \theta) = p(x_j^{(T)} | \ldots)\, p(x_j^{(T-1)} | x_j^{(T)}, \ldots)\, p(x_j^{(T-2)} | x_j^{(T-1)}, \cancel{x_j^{(T)}}, \ldots)$$

$$p(x_j^{(1)} | x_j^{(2)}, \cancel{x_j^{(3)}, \ldots, x_j^{(T)}}, \_\_\_)$$



$$q_{j,s,s'}^{(t)} = \underline{p(x_j^{(t)} = s', x_j^{(t-1)} = s | y^{(1)}, \ldots, y^{(t)}, X_{-j}, \theta)}$$

$$p(x_j^{(t)} = s | x_j^{(t+1)} = s', y^{(1)} \ldots y^{(t)}, \underline{\cancel{y^{(t+1)}, \ldots, y^{(T)}}}, X_{-j}, \theta) \propto q_{j,s,s'}^{(t+1)}$$

$$\begin{array}{c|ccc} & s' = S & I & R \\ \hline s = S & & & \\ I & & \boxed{q_{j,s,s'}^{(t)}} & \\ R & & & \end{array}$$

$$q_{j,s,s'}^{(t)} \propto p(x_j^{(t)} = s', x_j^{(t-1)} = s, y^{(t)} | y^{(1)}, \ldots, y^{(t-1)}, X_{-j}, \theta) =$$

$$= \underline{p(x_j^{(t)} = s' | x_j^{(t-1)} = s, \cancel{y^{(1..t-1)}}, X_{-j}, \theta)} \times \underline{p(y^{(t)} | x_j^{(t)} = s', \cancel{x_j^{(t-1)} = s, y^{(1..t-1)}}, X_{-j}, \theta)}$$

(iv)

$$\times\, p\left(x_j^{(t-1)}=s \mid y^{(1)},\dots,y^{(t-1)}, X_{-j}, \theta\right) = \sum_{s''} p\left(x_j^{(t-1)}=s,\ x_j^{(t-2)}=s'' \mid \dots\right)$$

$$= p\left(s' \mid s, X_{-j}, \theta\right) \times \mathrm{Binom}\left(y^{(t)} \mid I_{-j}^{(t)} + [s'=I], p\right) \times$$

$$\times \sum_{s''} q_{j,s'',s}^{(t-1)}$$

Stoch. Vit. alg.

- слева направо:

$$Q_j^{(1)}, Q_j^{(2)}, \dots, Q_j^{(T)}, \text{ где } Q_j^{(t)} = \begin{array}{c|c} & s' \\ \hline s & \begin{pmatrix} & | & \\ - & q_{j,s,s'}^{(t)} & - \\ & | & \end{pmatrix} \end{array}$$

- справа налево:

$$p\left(x_j^{(t)}=s \mid x_j^{(t+1)}=s', X_{-j}, \theta, y\right) \propto q_{j,s,s'}^{(t+1)}$$

Gibbs sampling:

- loop
  - for $j=1..N$
    - $\bar{x}_j \sim \mathrm{StochVit}(X_{-j})$

$$\bar{\pi} = (\pi, \dots, \pi)$$

$$p(\theta \mid y, x) \propto \underline{p(\theta)}\, p(x, y \mid \theta) = p(\beta)\, p(p)\, p(\mu)\, p(\bar{\pi})\, p(x, y \mid \theta)$$

$$\mathrm{Beta}(p \mid a_p, b_p) \propto p^{a_p-1}(1-p)^{b_p-1}$$

$$\theta \longrightarrow \max$$

$$\log p(\theta \mid y, x) = \mathrm{const} + (a_p-1)\log p + (b_p-1)\log(1-p) +$$

$$+ (a_\beta-1)\log\beta + (b_\beta-1)\log(1-\beta) + (a_\mu-1)\log\mu + (b_\mu-1)\log(1-\mu)$$

$$+ (a_\pi-1)\log\pi + (b_\pi-1)\log(1-\pi) +$$

$$+ \sum_{i=1}^{N}\left([x_i^{(1)}=I]\log\pi + [x_i^{(1)}=S]\log(1-\pi)\right) +$$

$$+ \sum_{t=1}^{T}\left(y^{(t)}\log p + (I^{(t)}-y^{(t)})\log(1-p)\right) +$$

$$\cancel{\sum_{t=1}^{T-1}\sum_{i=1}^{N}[x_i^{(t)}=R]} \cdot \log 1 +$$

$$+\sum_{t=1}^{T-1}\sum_{i=1}^{N}[x_i^{(t)}=I]\cdot\left([x_i^{(t+1)}=R]\log\mu+[x_i^{(t+1)}=I]\log(1-\mu)\right)+$$

$$+\sum_{t=1}^{T-1}\sum_{i=1}^{N}[x_i^{(t)}=S]\cdot\Big([x_i^{(t+1)}=S]\cdot I^{(t)}\cdot\log(1-\beta)+$$

$$+[x_i^{(t+1)}=I]\cdot\big(\underset{\#\text{ конт.}\to\text{заразился}}{P_i^{(t)}}\log\beta+\underset{\#\text{ конт.}\to\text{не зар.}}{N_i^{(t)}}\log(1-\beta)\big)\Big)\Big)$$

presence-only data

$$P_i^{(t)}+N_i^{(t)}=I^{(t)},\quad x_i^{(t+1)}=I\Rightarrow P_i^{(t)}\geq 1$$

$$\mathbb{E}\left[P_i^{(t)}\,\middle|\,x_i^{(t+1)}=I\right]=I^{(t)}\cdot p\left(\begin{matrix}\text{заражение}\\ \text{от 1 конт.}\end{matrix}\,\middle|\,\begin{matrix}\geq 1\\ \text{зараж.}\end{matrix}\right)=I^{(t)}\cdot\frac{\beta}{1-(1-\beta)^{I^{(t)}}}$$

$$P_i^{(t)}:=I^{(t)}\cdot\frac{\beta}{1-(1-\beta)^{I^{(t)}}},\quad N_i^{(t)}:=I^{(t)}-P_i^{(t)}$$

$$a_\pi'=a_\pi+\sum_{i=1}^{N}[x_i^{(1)}=I]=a_\pi+I^{(1)},\quad b_\pi'=b_\pi+\sum_{i=1}^{N}[x_i^{(1)}=S]=b_\pi+S^{(1)}$$

$$a_p'=a_p+\sum_{t=1}^{T}y^{(t)}\qquad b_p'=b_p+\sum_{t=1}^{T}(I^{(t)}-y^{(t)})$$

$$a_\mu'=a_\mu+\sum_{t=1}^{T-1}\sum_{i=1}^{N}[x_i^{(t)}=I,x_i^{(t+1)}=R]\qquad b_\mu'=b_\mu+\sum_{t=1}^{T-1}\sum_{i=1}^{N}[x_i^{(t)}=I,x_i^{(t+1)}=I]$$

$$a_\beta'=a_\beta+\sum_{t=1}^{T-1}\sum_{i=1}^{N}P_i^{(t)}\cdot[x_i^{(t)}=S,x_i^{(t+1)}=I],$$

$$b_\beta'=b_\beta+\sum_{t=1}^{T-1}\sum_{i=1}^{N}\left([x_i^{(t)}=S,x_i^{(t+1)}=S]\cdot I^{(t)}+[x_i^{(t)}=S,x_i^{(t+1)}=I]\cdot N_i^{(t)}\right)$$

EM-algorithm:

```
- for $k=1, \_\_$
  - E-шаг: Gibbs sampling given $\theta_k$
    - loop $j=1,\dots,N,1,\dots,N, \_\_$
      - $\bar{x}_j \sim p(\bar{x}_j \mid X_{-j}, y, \theta_k)$:
        - for $t=1..T$: compute $Q_j^{(t)}$
        - for $t=T..1$: $x_j^{(t)} \sim p(x_j^{(t)} \mid x_j^{(t+1)}, y, X_{-j}, \theta)$
    - save $X_r$
  - M-шаг:
    - $\theta_{k+1} = \arg\max p(\theta_{k+1} \mid X, y)$
      - presence-only data
```

stoch. Viterbi alg.

SIR extensions

① SIR ⟶ SEIR exposed $S \to E \to I \to R$

SEIS $S \to E \to I \to S$

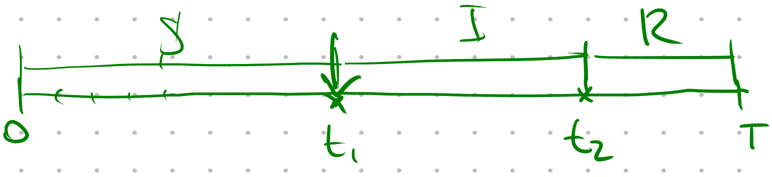
SEIRS $S \to E \to I \to R \to S$

SICS - carrier

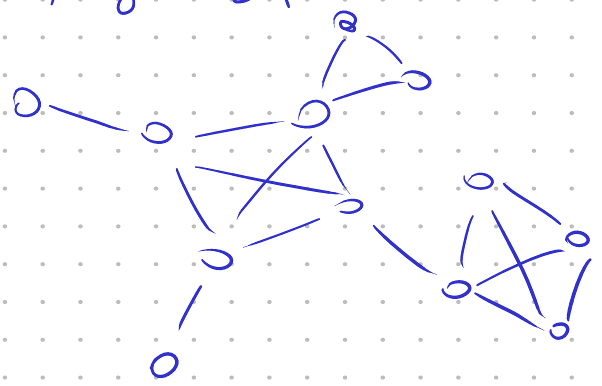
$$\begin{tikzcd} & & C \arrow[dl, shift left] \\ S \arrow[r] & I \arrow[ur, shift left] \arrow[dr] & \\ & & S \end{tikzcd}$$

$$\underline{\underline{R_0}} = \overset{\text{вирус}}{\beta} \cdot \underset{\text{сообщество}}{E[\#\text{ контактов}]}$$

② Непрерывное время



③ Модель заражения



$$(1-\beta)^{I^{(t)}}$$

$$(1-\beta)^{|I^{(t)} \cap Nei(x_i)|}$$