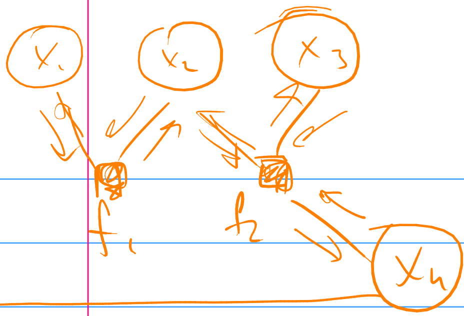
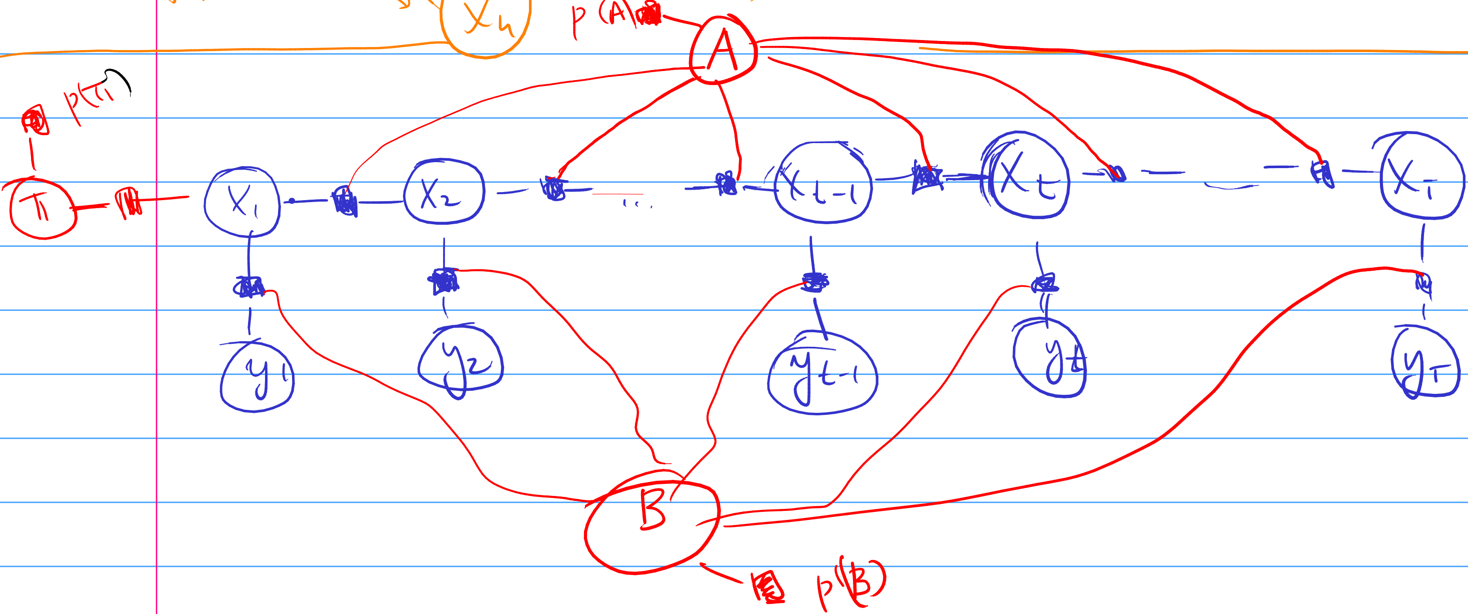


$f(x_1 \dots x_4) = f_1(x_1, x_2)\, f_2(x_2, x_3, x_4)$

$f(x_i) = \int f(x_1 \dots x_n)\, dx_1 \dots dx_{i-1}\, dx_{i+1} \dots dx_n$



[Sampling] $\bar{x} \sim p(\bar{x})$

$\mathbb{E}_{p(x)}[f(\bar{x})] \approx \frac{1}{R} \sum_{r=1}^{R} f(\bar{x}^{(r)})$

$\bar{x}^{(r)} \sim p(\bar{x})$

$p^*(\bar{x}) \propto p(\bar{x})$

$p(\theta | D) \propto p(\theta) p(D | \theta)$

$p(\theta | D) = \frac{p(\theta) p(D|\theta)}{p(D)}$

posterior

$= p(\theta) \prod_n p(d_n | \theta)$

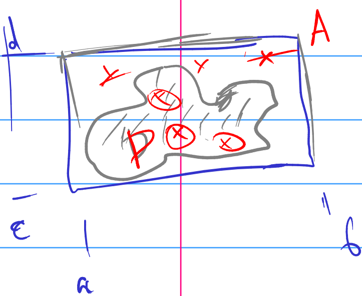
$p(x | D) = \int p(x|\theta) p(\theta|D)\, d\theta =$

$= \mathbb{E}_{p(\theta|D)}[p(x|\theta)]\, d\theta \approx$

$\approx \frac{1}{R} \sum_{r=1}^{R} p(x | \theta^{(r)})$

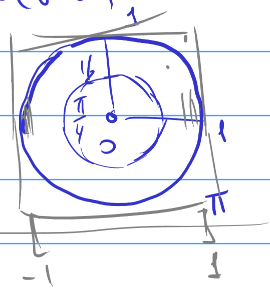
rand() $x \in \mathrm{Unif}[0,1]$

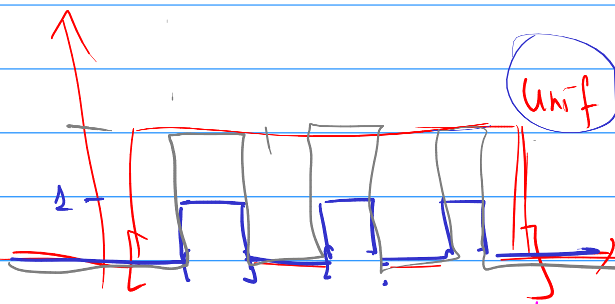
$\mathrm{Unif}([a,b)) \sim a + x(b-a)$



(x, y)

$(x, y) \sim \mathrm{Unif}$





$\bar{x} \sim \mathrm{Unif}(A) = 1/|A|$

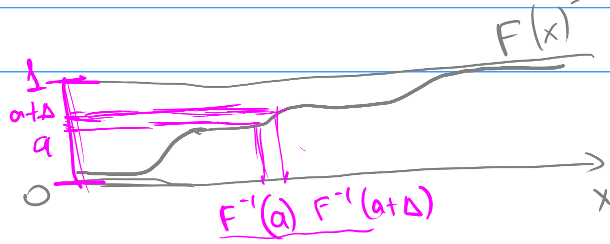
$p(\bar{x} | \bar{x} \in D) = \frac{p(\bar{x} | A)}{p(\bar{x} \in D)} = \frac{1/|A|}{|D|/|A|} = \frac{1}{|D|}$

$\mathrm{Unif}[0,1]$

$x \sim p(x)$

$F(x) = \int_{-\infty}^{x} p(a)\, da$

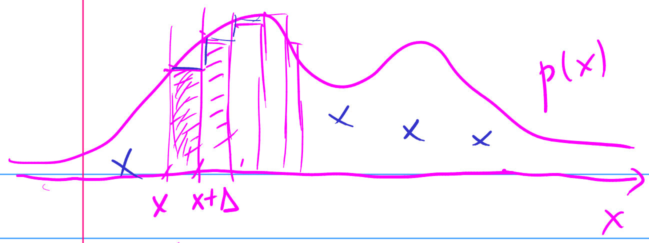
$p(F^{-1}(a)) = \lim_{\Delta \to 0} \frac{F^{-1}(a+\Delta) - F^{-1}(a)}{\Delta} = 1/p(a)$



$p(F^{-1}(x) \in [F^{-1}(a), F^{-1}(a+\Delta)]) = \Delta$

Box-Müller

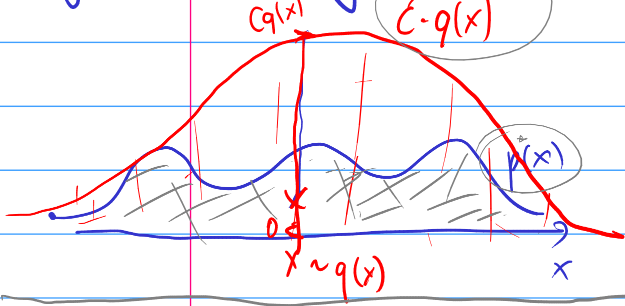
$N(x | 0, 1)$



$x \sim p(x) \iff (x, y) \sim \mathrm{Unif}(\text{график } p(x))$

$$\int_x^{x+\Delta} p(y)\,dy = F(x+\Delta) - F(x) \approx p(x)\cdot\Delta \approx p(x+\Delta)\cdot\Delta$$

Rejection sampling

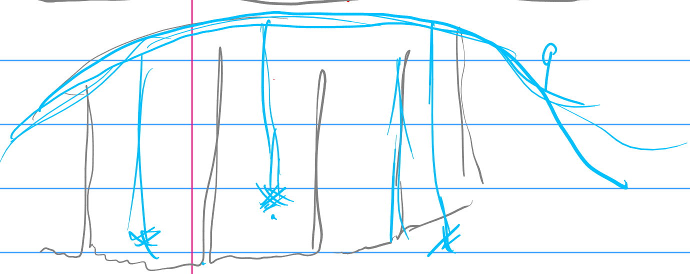


- $\bar{x} \sim q(\bar{x})$
- $y \sim \mathrm{Unif}[0, c\cdot q(\bar{x})]$
- if $y < \tilde{p}(\bar{x})$ accept $\bar{x}$

  else reject $\bar{x}$

$(\bar{x}, y)$

$p^*(\bar{x}) \propto p(\bar{x})$



Importance sampling

$\bar{x}^{(r)} \sim p(\bar{x})$

$\bar{x} \sim q(\bar{x})$

$$E_p[f] \approx \frac{1}{R}\sum_{r=1}^{R} f(\bar{x}^{(r)})$$

$p(\bar{x}) > 0 \Rightarrow q(\bar{x}) > 0$

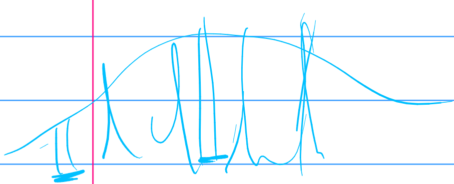
importance weights

$$\approx \frac{1}{R}\sum f(x^{(r)})\frac{p(x^{(r)})}{q(x^{(r)})}$$

$$E_p[f] = \int f(\bar{x})p(\bar{x})\,d\bar{x} = \int f(x)\,p(\bar{x})\frac{q(\bar{x})}{q(\bar{x})}\,dx = E_{q(\bar{x})}\left[f(\bar{x})\cdot\frac{p(\bar{x})}{q(\bar{x})}\right]$$

$$p(\bar{x}) = \frac{1}{Z_p}p^*(\bar{x}); \quad Z_p = \int p^*(\bar{x})\,d\bar{x} = \int \frac{p^*(\bar{x})}{q(\bar{x})}q(\bar{x})\,d\bar{x} = E_q\left[\frac{p^*(\bar{x})}{q(\bar{x})}\right]$$

$$E_p[f] = E_q\left[f\cdot\frac{p}{q}\right] = E_q\left[f\cdot\frac{p^*}{q}\right] \Big/ E_q\left[p^*/q\right] = \frac{E_q\left[f\cdot\frac{p^*}{q^*}\right]}{E_q\left[p^*/q^*\right]}$$



MCMC Markov Chain Monte-Carlo

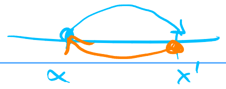
$X_1, X_2, \dots, X_t, \dots$ $\qquad \pi(\bar x) = \int T(\bar x, \bar x')\pi(\bar x')\,d\bar x'$ $\qquad \pi(\bar x) = p(\bar x)$

stationary distribution

условие баланса

$$\boxed{\pi(x)\,p(x'|x) = \pi(x')\cdot p(x|x')}$$

$q(\bar x_t|\bar x_{t-1})$



$\Rightarrow \pi$ - стацион. распред.

$$\pi(x)\,p(x'|x) = \pi(x')\,p(x|x')$$

$$\pi(x)\cdot\int p(x'|x)\,dx' = \pi(x) \stackrel{?}{=} \int p(x|x')\,\pi(x')\,dx'$$

$$\| \quad \int \pi(x)\,p(x'|x)\,dx'$$

Metropolis-Hastings

$x \sim p(\bar x)$ $\qquad q(\bar x|\bar x^{(t)})$

Переход из $\bar x^{(t)}$:

- $\bar x \sim q(\bar x|\bar x^{(t)})$
- $a = \frac{p(\bar x)}{p(\bar x^{(t)})}\cdot\frac{q(\bar x^{(t)}|\bar x)}{q(\bar x|\bar x^{(t)})}$
- if $a \geq 1$ accept $\bar x^{(t+1)} = \bar x$

else $\bar x^{(t+1)} = \begin{cases}\bar x & \text{с вер. } a \\ \bar x^{(t)} & \text{с вер. } 1-a\end{cases}$

$$p(\bar x)\cdot p_t(\bar x'|\bar x) = p(\bar x')\,p_t(\bar x|\bar x')$$

$$a(\bar x'|\bar x) = \frac{p(\bar x')}{p(\bar x)}\cdot\frac{q(\bar x|\bar x')}{q(\bar x'|\bar x)} = \frac{1}{a(\bar x|\bar x')}$$

пусть $a(\bar x'|\bar x) \geq 1$

Тогда $p_t(\bar x'|\bar x) = q(\bar x'|\bar x)$

$$p_t(\bar x|\bar x') = q(\bar x|\bar x')\cdot\frac{p(\bar x)}{p(\bar x')}\cdot\frac{q(\bar x'|\bar x)}{q(\bar x|\bar x')}$$

$$p(\bar x)\,q(\bar x'|\bar x) = \cancel{p(\bar x')}\cdot \cancel{q(\bar x|\bar x')}\,\frac{p(\bar x)}{\cancel{p(\bar x')}}\,\frac{q(\bar x'|\bar x)}{\cancel{q(\bar x|\bar x')}}$$

Gibbs sampling

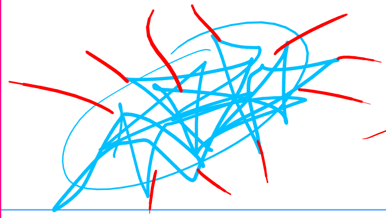
MCMC

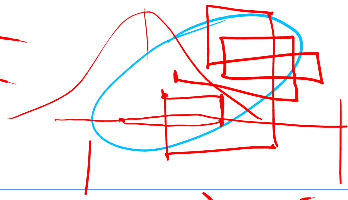
- $\bar x^{(0)}$
- for $t = 0, \dots$
  - for $i = 1..n$:

    $\left(x_i^{(t+1)} \sim p(x_i|x_1^{(t+1)} \dots x_{i-1}^{(t+1)}, x_{i+1}^{(t)}, \dots, x_n^{(t)})\right)$

$p(\bar x) = p(x_1, x_2, \dots, x_n)$

$p(x_k|x_1 \dots x_{k-1}, x_{k+1}, \dots, x_n)$





$\bar{x}_{-i} = (x_1 \ldots x_{i-1}, x_{i+1} \ldots x_n)$

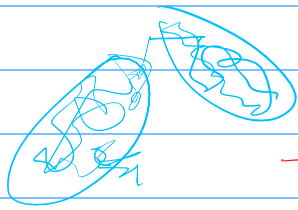
$$q(\bar{x}'|\bar{x}) = \begin{cases} p(x_i'|\bar{x}_{-i}), & \bar{x}_{-i} = \bar{x}'_{-i} \\ 0, & \bar{x}' \text{ и } \bar{x} \text{ отлич. не только в } i\text{-й координате} \end{cases}$$

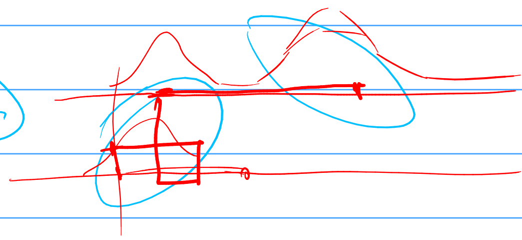
$\bar{x} \to \bar{x}'$

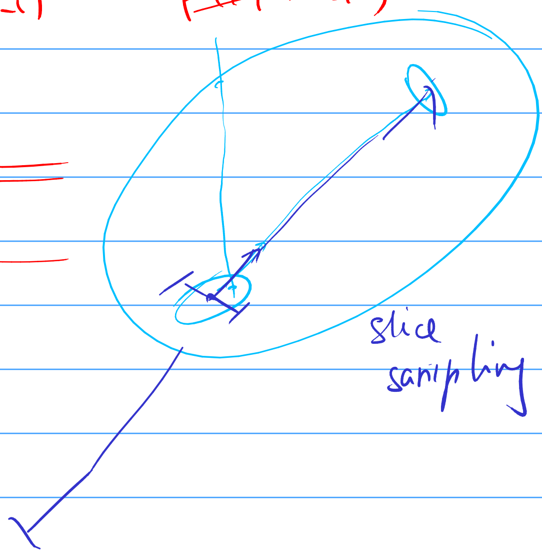
$\bar{x}_{-i} = \bar{x}'_{-i}$

$$a = \frac{p(\bar{x}')}{p(\bar{x})} \cdot \frac{q(\bar{x}|\bar{x}')}{q(\bar{x}'|\bar{x})} = \frac{p(\bar{x}')}{p(\bar{x})} \cdot \frac{p(x_i|\bar{x}_{-i})}{p(x_i'|\bar{x}_{-i})} =$$

$$= \frac{p(x_i'|\cancel{\bar{x}_{-i}})\, p(\bar{x}_{-i})}{p(x_i|\cancel{\bar{x}_{-i}})\, \cancel{p(\bar{x}_{-i})}} \cdot \frac{p(\cancel{x_i}|\bar{x}_{-i})}{\cancel{p(x_i'|\bar{x}_{-i})}} = 1$$







slice sampling