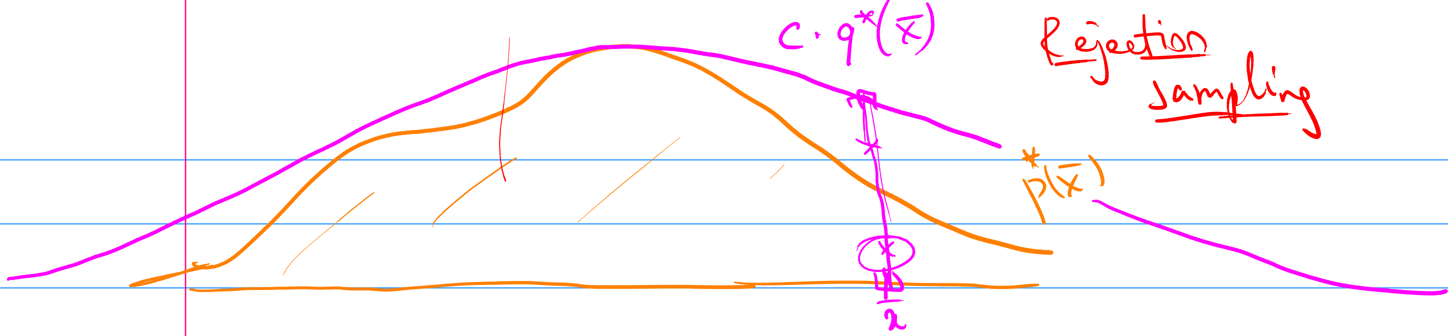
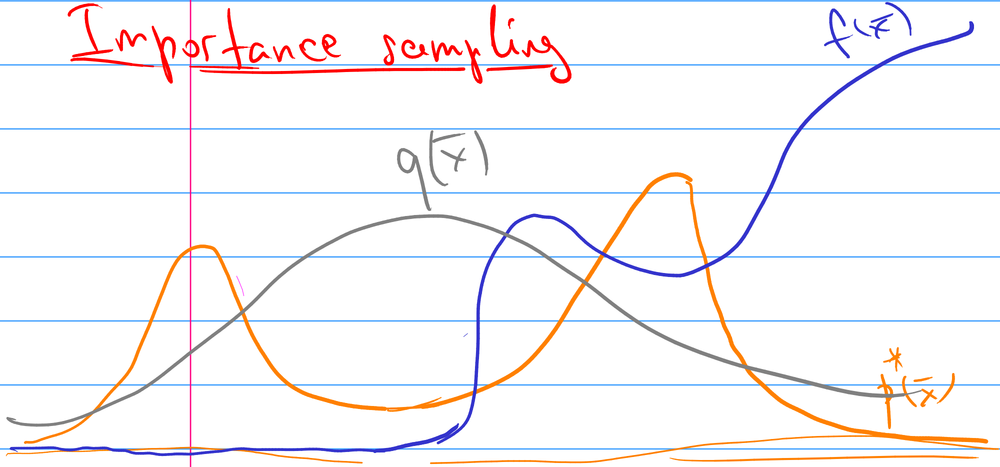
Rejection sampling



Importance sampling

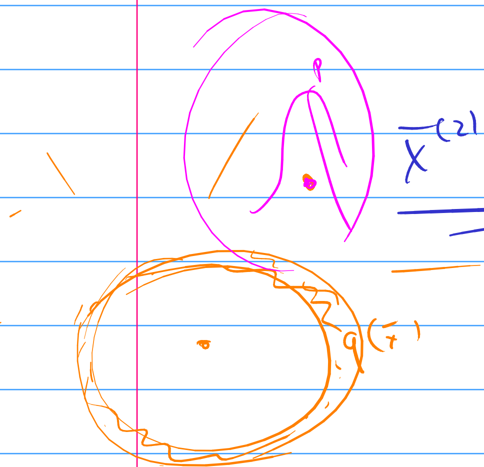


$$\mathbb{E}_{p(\bar{x})}[f(\bar{x})] \approx$$

$$\approx \frac{1}{R} \sum_{1}^{R} f(\bar{x}^{(r)})$$

$$\mathbb{E}_{p(\bar{x})}[f(\bar{x})] = \int f(\bar{x}) p(\bar{x}) d\bar{x} =$$

$$= \int f(\bar{x}) \frac{p(\bar{x})}{q(\bar{x})} q(\bar{x}) dx = \mathbb{E}_{q(\bar{x})}\left[f(\bar{x}) \cdot \frac{p(\bar{x})}{q(\bar{x})}\right] \approx$$



$\bar{x}^{(r)} \sim q(\bar{x})$

$$\approx \frac{1}{R} \sum_{1}^{R} \frac{p(\bar{x}^{(r)})}{q(\bar{x}^{(r)})} f(\bar{x}^{(r)})$$

importance weights

$$p(\bar{x}) = \frac{1}{Z_p} p^*(\bar{x})$$

$$q(\bar{x}) = \frac{1}{Z_q} q^*(\bar{x})$$

$$Z_q = \frac{q^*}{q}$$

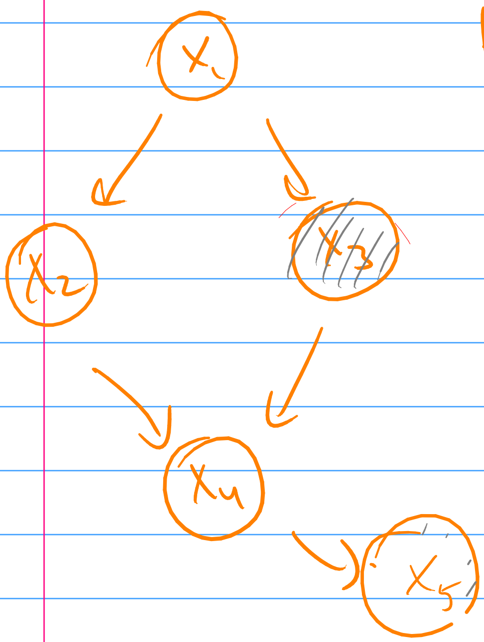
$$\mathbb{E}_p[f] = \int p \cdot f \, d\bar{x} = \int f \cdot \frac{p}{q} \cdot q \, d\bar{x} =$$

$$= \int f \frac{p^*/Z_p}{q^*/Z_q} \cdot q \, d\bar{x} \approx \frac{1}{R} \sum \left[\frac{Z_q}{Z_p} \cdot \frac{p^*}{q^*}\right] f(x^{(r)})$$

$$\frac{Z_p}{Z_q} = \int \frac{p^*(\bar{x})}{Z_q} d\bar{x} = \int p^*(\bar{x}) \frac{q(\bar{x})}{q^*(\bar{x})} d\bar{x} =$$

$$= \mathbb{E}_q\left[\frac{p^*(\bar{x})}{q^*(\bar{x})}\right] \approx$$

$$\approx \frac{1}{R}\sum_{1}^{R} \frac{p^*(x^r)}{q^*(x^r)}$$



$$\underline{p(x_1, \dots, x_5)} = p(x_1)\,p(x_2|x_1)\,p(x_3|x_1)\cdot$$
$$\cdot\, p(x_4|x_2,x_3)\,p(x_5|x_4)$$

$$p(x_1, x_2, x_4, x_5 \mid \underline{x_3})$$

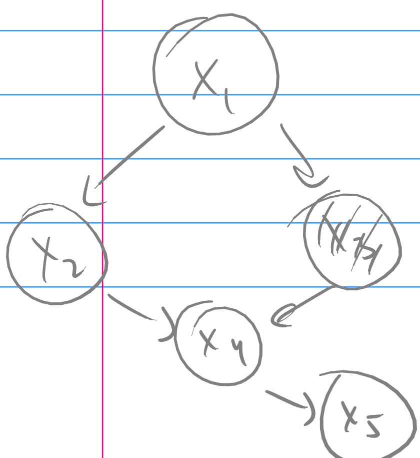
Rejection sampling:

$$\boxed{(x_1 \dots x_5)}$$

Importance Sampling $\quad \mathbb{E}_p[f(\bar{x})]$

$q(\bar{x}) = \underline{x_3}$ - fix, $\underline{x_1 \dots x_5 \sim \text{Unif}}$ $\qquad q(\underline{\bar{x}}) = \frac{1}{Z}$

$$\mathbb{E}_p[f(\bar{x})] \approx \frac{1}{R}\sum_{1}^{R}\left[\frac{p(\bar{x}^r)}{q(\bar{x}^r)}\right]\cdot f(\bar{x}^r)$$



q: $\quad x_1 \sim p(x_1) \qquad \underline{x_3 = \text{fix}}$

$x_2 \sim p(x_2|x_1)$

$x_4 \sim p(x_4|x_2,x_3)$

$x_5 \sim p(x_5|x_4)$

$(x_1, \ldots, x_4)$ $\quad p(x_1 - x_5) = p(\cancel{x_1})\, p(x_2|\cancel{x_1})\, p(x_3|x_2)$

$q(x_1 - x_5) = p(\cancel{x_1})\, p(x_2|\cancel{x_1})\, p(x_4|x_2, x_3)\, p(x_5|x_4)$

evidence

$p = p(x|\underline{y})$ $\qquad \dfrac{p(\bar{x})}{q(\bar{x})} = p(x_3|x_2)$

likelihood weighted sampling

$$E_p[f] \approx \frac{1}{R} \sum_{r=1}^{R} \left(f(x^{(r)}) \cdot \prod_{x_i \in y} p(x_i | par(x_i)) \right)$$

MCMC - Markov Chain Monte Carlo

$X_1, X_2, \ldots$

$p(X_n | X_{n-1}, \ldots, X_1) = p(X_n | X_{n-1})$

$p^{(0)}(\bar{x}) \qquad p^{(1)}(\bar{x}') = \int T(\bar{x}'; \bar{x})\, p^{(0)}(\bar{x})\, d\bar{x}$

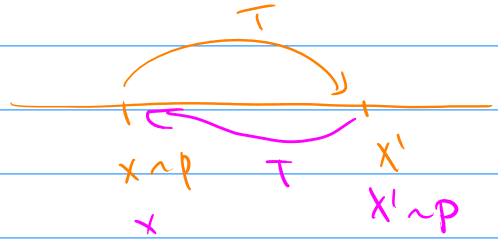
$p^{(1)}(\bar{x}) = A\, p^{(0)}(\bar{x})$

$p^{(k)}(\bar{x}) = A\, p^{(k-1)}(\bar{x}) \qquad p^{(k)}(\bar{x}') = \int T(\bar{x}'; \bar{x})\, p^{(k-1)}(\bar{x})\, d\bar{x}$

$p^{(0)}, p^{(1)}, p^{(2)}, \ldots \longrightarrow \underline{\pi(\bar{x}')} = \int T(\bar{x}', \bar{x})\, \underline{\pi(\bar{x})}\, d\bar{x}$

стацион./инвар. распр.

Условие баланса: $\forall\, x, x' \quad T(x', x)\, p(x) = T(x, x')\, p(x')$ $\Rightarrow$ p – стац. р.

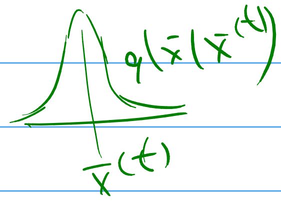


$$\pi(x') \overset{?}{=} \int T(x', x)\, \pi(x)\, dx = \int T(x, x')\, \pi(x')\, dx = \pi(x') \cdot 1$$

Metropolis–Hastings

$\boxed{p(\bar{x})}$

$q(\bar{x} \mid \bar{x}^{(t)})$ - sampler



Переход MC:

- $\bar{x}' \sim q(\bar{x}' \mid \bar{x}^{(t)})$
- $$a(\bar{x}', \bar{x}^{(t)}) = \frac{p(\bar{x}')}{p(\bar{x}^{(t)})} \cdot \frac{q(\bar{x}^{(t)} \mid \bar{x}')}{q(\bar{x}' \mid \bar{x}^{(t)})} = \frac{p^*(\bar{x}')}{p^*(\bar{x}^{(t)})} \cdot \frac{q(\bar{x}^{(t)} \mid \bar{x}')}{q(\bar{x}' \mid \bar{x}^{(t)})}$$

$$\boxed{a(\bar{x}', \bar{x}) = \frac{1}{a(\bar{x}, \bar{x}')}}$$

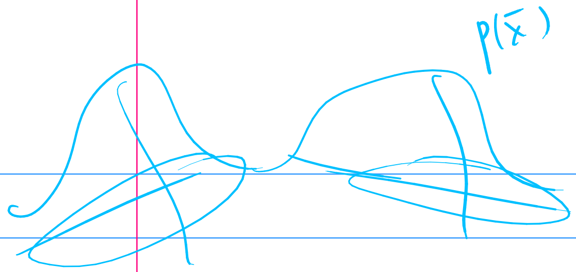
- if $a \geq 1$ then $\bar{x}^{(t+1)} = \bar{x}'$

  else $\bar{x}^{(t+1)} = \begin{cases} \bar{x}' & \text{с вер. } a \\ \bar{x}^{(t)} & \text{с вер. } 1-a \end{cases}$

$$T(x, x')\, p(x') \overset{?}{=} T(x', x)\, p(x) \qquad a(\bar{x}, \bar{x}') \leq 1$$

$$\| $$

$$\cancel{q(\bar{x} \mid \bar{x}')} \cdot \frac{p(\bar{x})\; q(\bar{x}' \mid \bar{x})}{\cancel{p(\bar{x}')}\; \cancel{q(\bar{x} \mid \bar{x}')}} \cdot \cancel{p(\bar{x}')} \overset{?}{=} p(\bar{x}) \cdot q(\bar{x}' \mid \bar{x})$$



$p(\bar{x})$

$$p(\bar{\theta}|D) \propto p(\theta) \prod_n p(\bar{x}_n|\theta)$$

$$\boxed{\log p^*(\bar{\theta}|D) = \log p(\bar{\theta}) + \sum_n \log p(\bar{x}_n|\bar{\theta})}$$