- ~~Дерево~~
- Фактор-граф простые

$\Rightarrow$ Алгоритм передачи сообщений

- Не дерево

Approximate inference

- Более сложные факторы

Expectation Propagation

$q(\theta) \approx p(\theta|D)$

Variational approximations

sampling

$p(\bar{x})$ $\quad \bar{z} \sim \mathrm{Unif}([0,1]) \quad \bar{z} \to \boxed{\quad} \to \bar{x} \sim p(\bar{x})$ $\Leftarrow$ умеем считать или $p^*(x) \propto p(x)$

1) $$p(\bar{x}|D) = \int p(\bar{x}|\bar{\theta})\, \underline{p(\bar{\theta}|D)}\, d\bar{\theta} = \mathbb{E}_{p(\theta|D)}\left[p(\bar{x}|\bar{\theta})\right]$$

$$\bar{\theta}^{(r)} \sim p(\bar{\theta}|D) \quad \underset{\Rightarrow}{\propto p(\bar{\theta})p(\bar{\theta}|D)} \quad \approx \frac{1}{R}\sum_{r=1}^{R} p(\bar{x}|\bar{\theta}^{(r)})$$

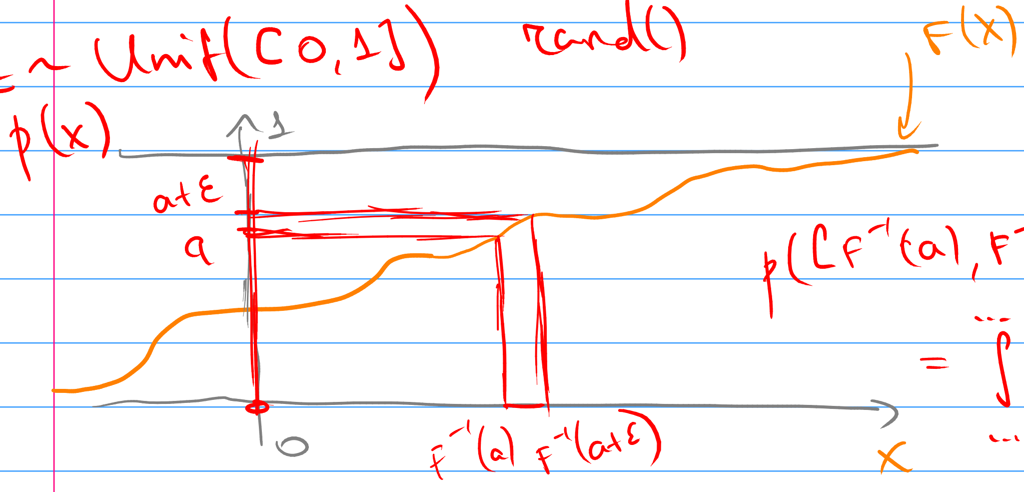
2) $$Q(\bar{\theta}, \bar{\theta}^{(m)}) = \int p(z|x, \theta^{(m)}) \ln p(x, z|\theta)\, dZ =$$

$$= \mathbb{E}_{p(z|x,\theta^{(m)})}\left[\ln p(x, z|\theta)\right] \approx \frac{1}{R}\sum_{z} \ln p(x, z^{(r)}|\bar{\theta})$$

Monte-Carlo EM

---

1) $z \sim \mathrm{Unif}([0, 1])$ $\quad$ rand()



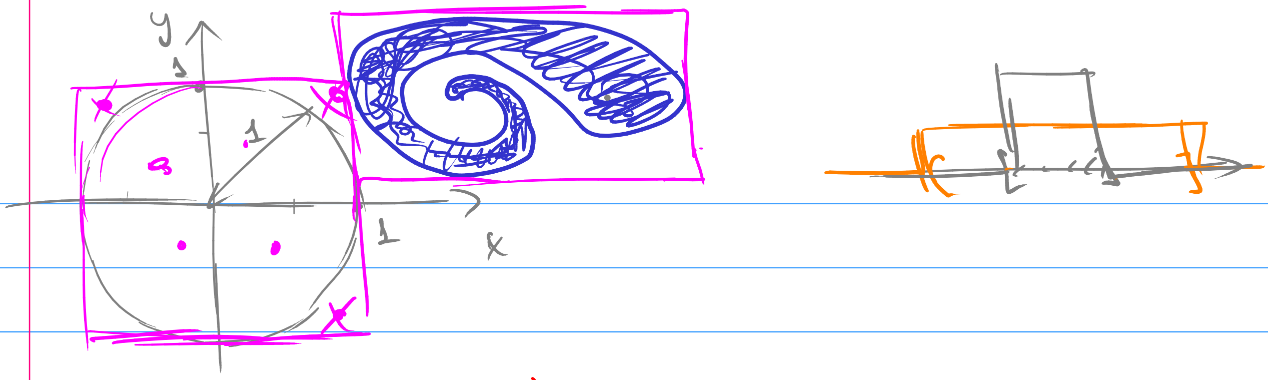
$$p([F^{-1}(a), F^{-1}(a+\varepsilon)]) =$$

$$= \int_{\dots}^{\dots} p(x)\, dx =$$

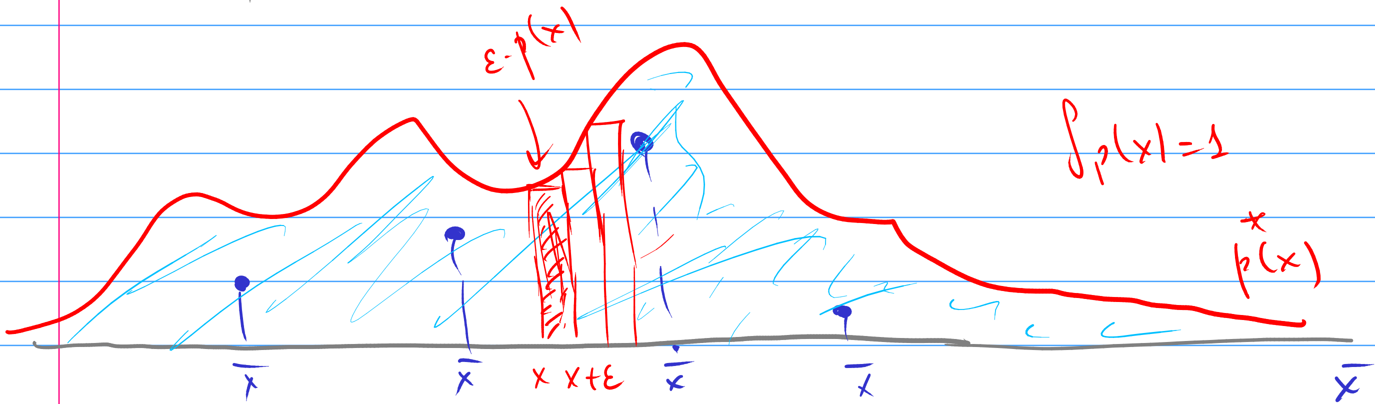
$$= F(\dots) - F(\dots) =$$

$$= \varepsilon$$

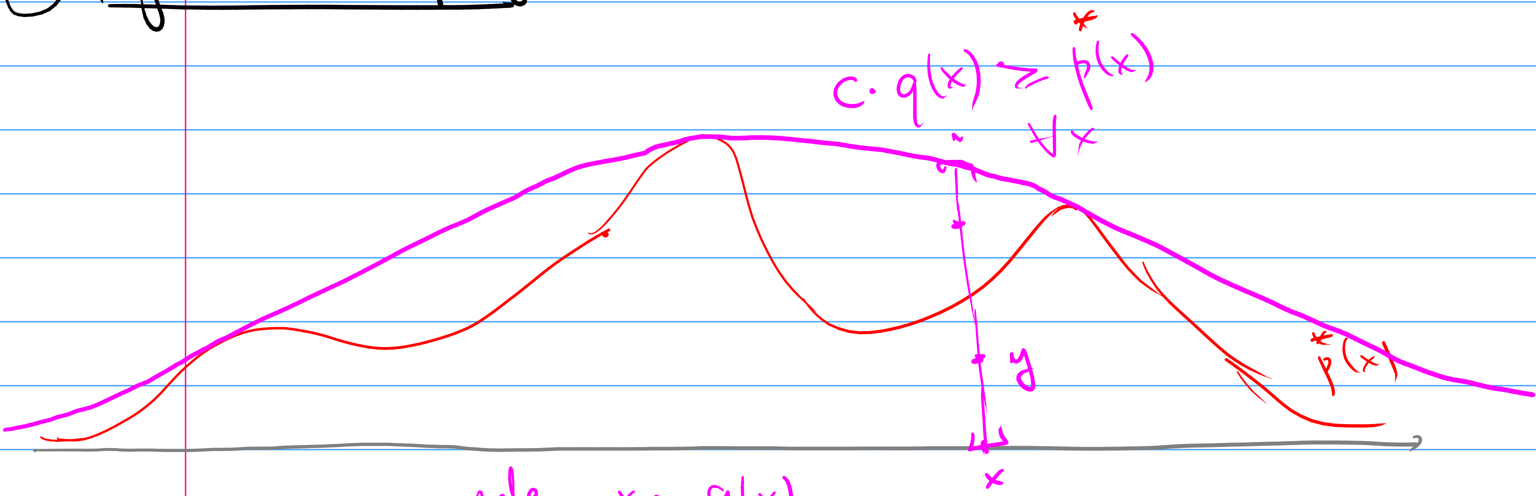
2)



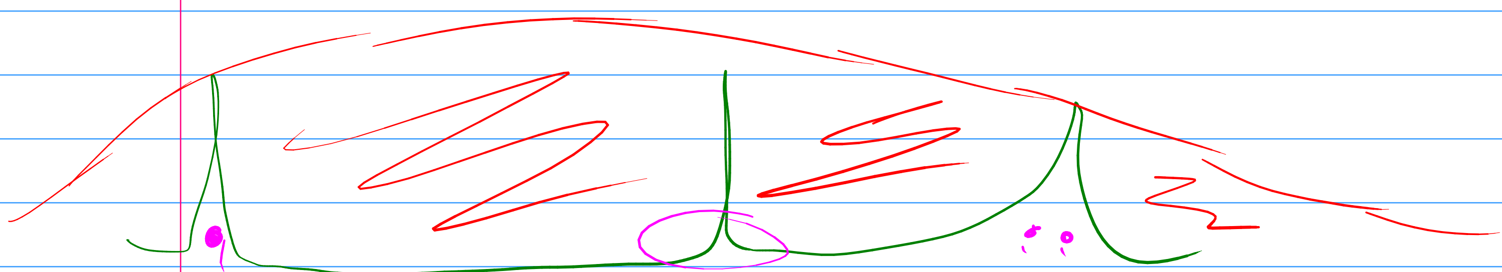
3)

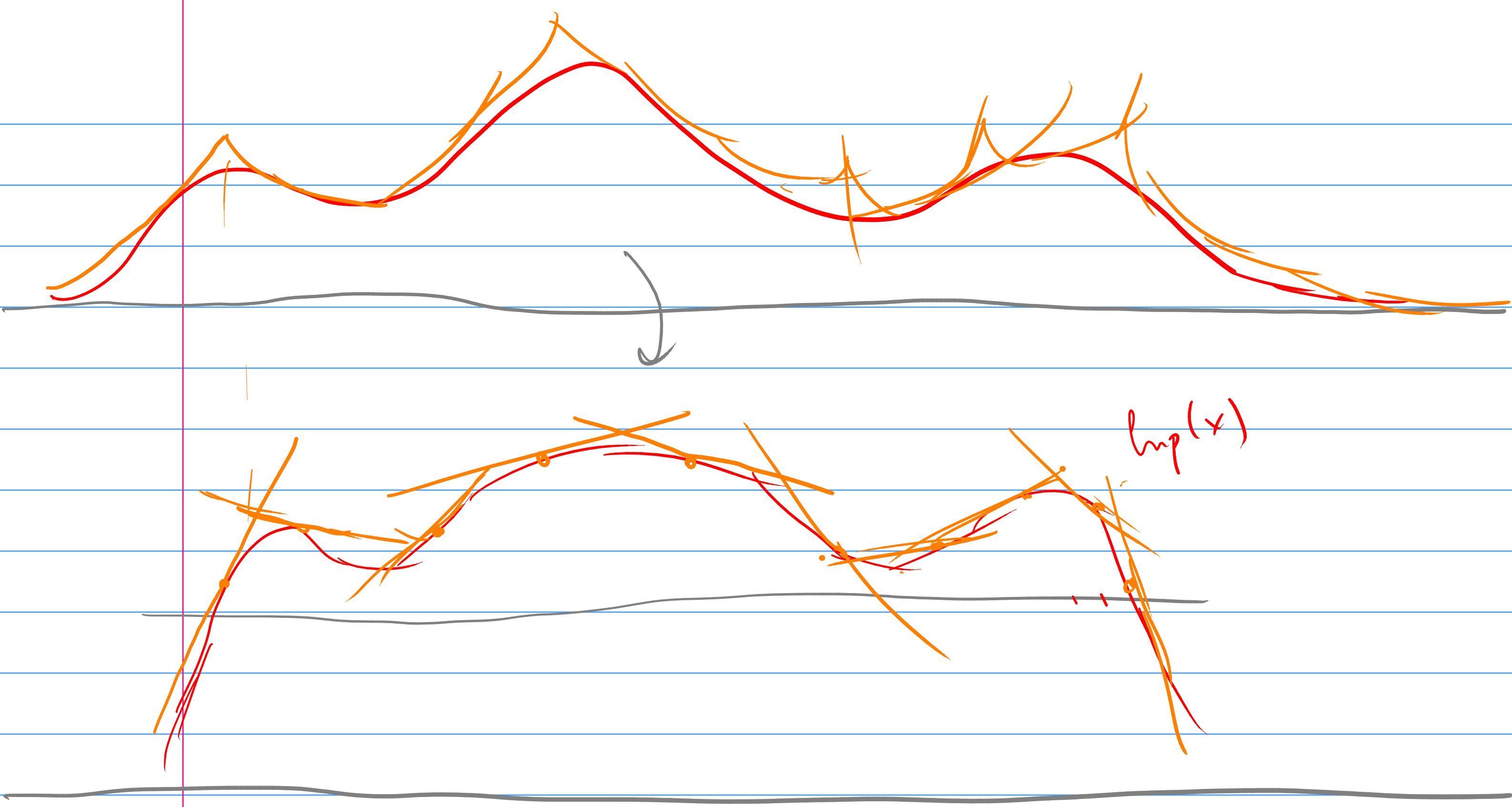


① Rejection sampling



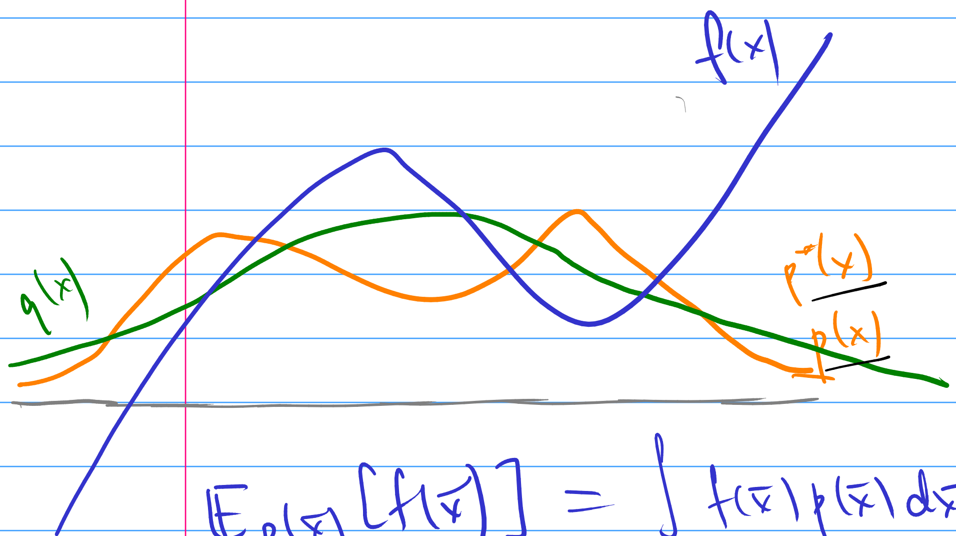
- sample $x \sim q(x)$
- sample $y \sim [0, c \cdot q(x)]$
- if $y > p(x)$ reject
  else accept x

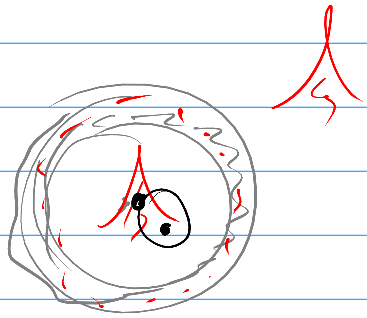




② Importance sampling

$$\mathbb{E}_{p(\bar{x})}[f(\bar{x})] = ?$$





$$\mathbb{E}_{p(\bar{x})}[f(\bar{x})] = \int f(\bar{x})\, p(\bar{x})\, d\bar{x} = \int f(\bar{x}) \frac{p(\bar{x})}{q(\bar{x})} q(\bar{x})\, d\bar{x} =$$

$$= \mathbb{E}_{q(\bar{x})}\left[f \cdot \frac{p}{q}\right] \overset{\bar{x}^{(r)} \sim q(\bar{x})}{\approx} \frac{1}{R} \sum_{r=1}^{R} f(\bar{x}^{(r)}) \frac{p(\bar{x}^{(r)})}{q(\bar{x}^{(r)})}$$

$$p(\bar{x}) = \frac{1}{Z_p} p^*(\bar{x})$$

$$q(\bar{x}) = \frac{1}{Z_q} q^*(\bar{x})$$

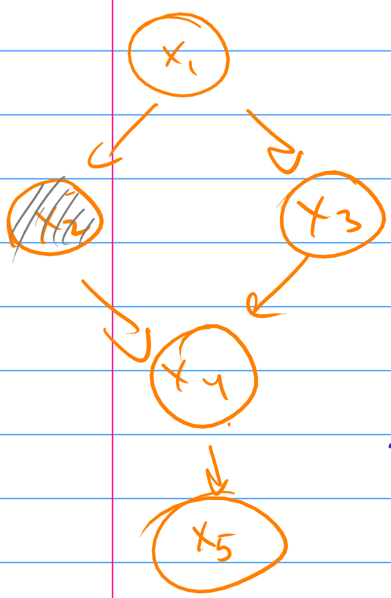
$$\mathbb{E}_p[f] =$$

$$= \mathbb{E}_q\left[f \cdot \frac{p}{q}\right] = \mathbb{E}_q\left[f \cdot \frac{p^*(\bar{x})}{q^*(\bar{x})} \cdot \frac{Z_q}{Z_p}\right]$$

importance weights

$$\frac{Z_p}{Z_q} = \int \frac{1}{Z_q} p^*(\bar{x})\,d\bar{x} = \int p^*(\bar{x}) \frac{q(\bar{x})}{q^*(\bar{x})}\,d\bar{x} =$$

$$= \mathbb{E}_q\left[\frac{p^*}{q^*}\right] \approx \frac{1}{R} \sum_{1}^{R} \frac{p^*(\bar{x}^{(r)})}{q^*(\bar{x}^{(r)})}$$



$$p(x_1 \ldots x_5) = p(x_1)\,p(x_2|x_1)\,p(x_3|x_1)\,p(x_4|x_2,x_3)\,p(x_5|x_4)$$

$x_1, x_3, x_4, x_5 \sim p(x_1, x_3, x_4, x_5 | x_2)$?

Rejection sampling: $x_1 \ldots x_5 \sim p(x_1 \ldots x_5)$

Importance sampling:

$$\mathbb{E}_p[f] = \mathbb{E}_q\left[f \cdot \frac{p}{q}\right] \approx \frac{1}{R} \sum_{1}^{R} \boxed{\frac{p(\bar{x}^{(r)})}{q(\bar{x}^{(r)})}}\, f(\bar{x}^{(r)})$$

q: $x_1 \sim p(x_1),\ x_3 \sim p(x_3|x_1),\ x_4 \sim p(x_4|x_2, x_3),\ x_5 \sim p(x_5|x_4)$

$$p(x_1, x_3, x_4, x_5 | x_2) \propto p(x_1, x_2, \ldots x_5) = p(x_1)\,p(x_2|x_1) \ldots$$

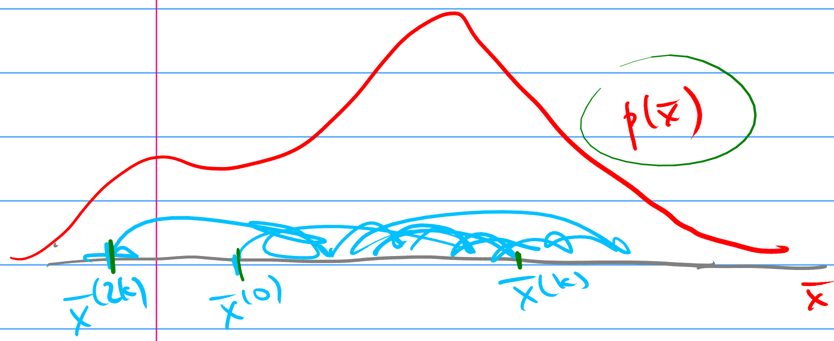
$$q(x_1 \cancel{x_2} \ldots x_5) = p(x_1)\,p(x_3|x_1) \ldots p(x_5|x_4)$$

$$\boxed{\mathbb{E}_p[f] \approx \frac{1}{R} \sum_{1}^{R} f(x^{(r)}) \cdot \prod_{x_i \in \text{evidence}} p(x_i | par(x_i))}$$

likelihood weighted sampling

MCMC — Markov Chain Monte Carlo

$$x_1, x_2, \ldots, x_k, \ldots$$
$$q(x_k \mid x_1 \ldots x_{k-1}) = q(x_k \mid x_{k-1})$$



$$q^{(0)}(\bar{x}), \quad q^{(1)}(\bar{x}) = \int T(\bar{x}; \bar{x}')\, q^{(0)}(\bar{x}')\, d\bar{x}'$$

$$q^{(1)}(\bar{x}) = A\, q^{(0)}(\bar{x})$$

$$q^{(0)}, q^{(1)}, q^{(2)}, \ldots \longrightarrow \underline{\pi(\bar{x}) = \int T(\bar{x}; \bar{x}')\, \pi(\bar{x}')\, d\bar{x}'}$$

стацион. распред.

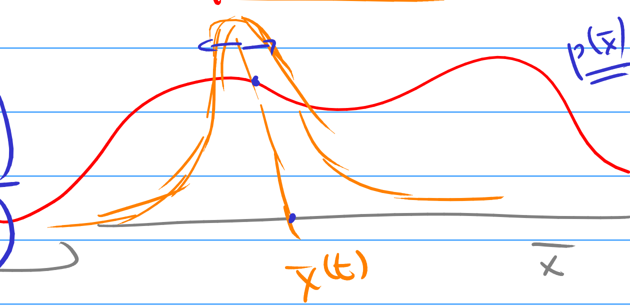
Условие баланса: если $\forall x, x'$ $\quad p(\bar{x})\, T(\bar{x}'; \bar{x}) = p(\bar{x}')\, T(\bar{x}; \bar{x}')$,
то $p(\bar{x})$ — стацион. распр. для T

$$p(\bar{x}) \overset{?}{=} \int T(\bar{x}; \bar{x}')\, p(\bar{x}')\, d\bar{x}' = \int p(\bar{x})\, T(\bar{x}'; \bar{x})\, d\bar{x}' = p(\bar{x})$$

Metropolis–Hastings algorithm

$\overset{*}{p}(\bar{x})$

sampler $\quad q(\bar{x} \mid \bar{x}^{(t)})$



Markov chain: $\quad \bar{x}^{(t)}$

- $\bar{x}' \sim q(\bar{x}' \mid \bar{x}^{(t)})$
- $a(\bar{x}', \bar{x}^{(t)}) = \dfrac{\overset{*}{p}(\bar{x}')}{\overset{*}{p}(\bar{x}^{(t)})} \cdot \dfrac{q(\bar{x}^{(t)} \mid \bar{x}')}{q(\bar{x}' \mid \bar{x}^{(t)})}$
- if $a \geq 1$ $\quad \bar{x}^{(t+1)} := \bar{x}^{(t)}$

  else $\quad \bar{x}^{(t+1)} := \begin{cases} \bar{x}' & \text{с вер-тью } a \\ \bar{x}^{(t)} & \text{с вер-тью } 1-a \end{cases}$

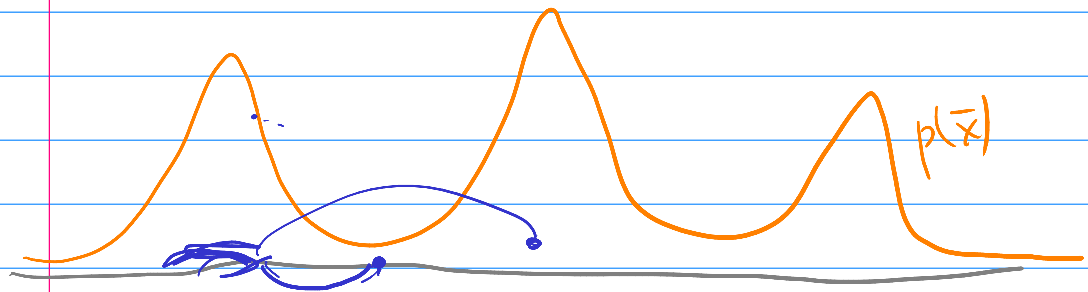
$$a(\bar{x}, \bar{x}') = \frac{1}{a(\bar{x}', \bar{x})}$$

$$p(\bar{x})\, \underbrace{T(\bar{x}';\bar{x})} \stackrel{?}{=} p(\bar{x}')\, T(\bar{x};\bar{x}')$$

$a(\bar{x}',\bar{x}) \geq 1$

$$p(\bar{x}) \cdot q(\bar{x}'|\bar{x}) \stackrel{?}{=} p(\bar{x}')\, q(\bar{x}|\bar{x}') \cdot a(\bar{x},\bar{x}')$$

$$\cancel{p(x)\, q(\bar{x}'|\bar{x})} \stackrel{?}{=} \cancel{p(\bar{x}')\, q(\bar{x}|\bar{x}')} \cdot \frac{p^*(\bar{x})}{p^*(\bar{x}')} \cdot \frac{\cancel{q(\bar{x}'|\bar{x})}}{\cancel{q(\bar{x}|\bar{x}')}}$$



$$\boxed{p(\bar{\theta}|D)} \propto p(\bar{\theta}) \cdot \prod_n p(\bar{x}_n|\bar{\theta})$$

Gibbs sampling $\qquad p(\bar{x}) = p(x_1, x_2, \ldots, x_n)$

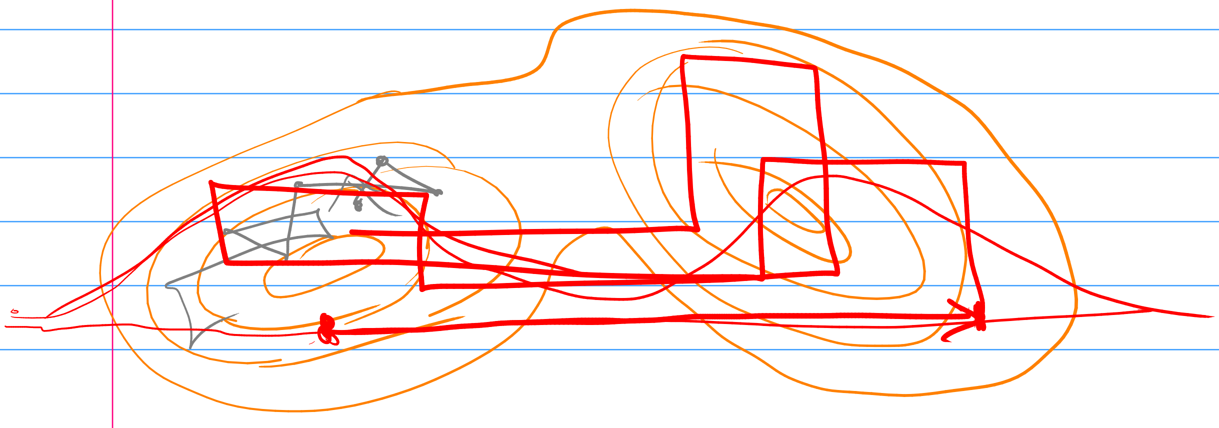
$x_1^{(1)}, \ldots, x_1^{(n)}$

$x_1^{(2)}, x_1^{(1)} \ldots x_1^{(n)}$

$(x_1^{(2)} \ldots x_{n-1}^{(2)}, x_n^{(2)}) \to \bar{x}^{(2)}$

- loop over k:
  - loop over $i = 1..n$:

$$x_i^{(k+1)} \sim p(x_i^{(k+1)} | x_1^{(k+1)}, \ldots, x_{i-1}^{(k+1)}, x_{i+1}^{(k)}, \ldots, x_n^{(k)})$$



$(\ldots \boxed{x_i} \ldots)$

$(\ldots \boxed{x'_i} \ldots)$

$$q(\bar{x}' | \bar{x}) = \begin{cases} 0, & \bar{x}'_{-i} \neq \bar{x}_{-i} \\ p(x'_i | \bar{x}_{-i}), & \bar{x}'_{-i} = \bar{x}_{-i} \end{cases}$$

$\ll \bar{x}'_{-i}$

$$a(\bar{x}', \bar{x}) = \frac{p(\bar{x}')}{p(\bar{x})} \frac{q(\bar{x} | \bar{x}')}{q(\bar{x}' | \bar{x})} = \frac{p(\bar{x}')}{p(\bar{x})} \cdot \frac{p(x_i | \bar{x}_{-i})}{p(x'_i | \bar{x}_{-i})} =$$

$$= \frac{p(\bar{x}'_i | \bar{x}'_{-i})\, p(\bar{x}'_{-i})}{p(\bar{x}_i | \bar{x}_{-i})\, p(\bar{x}_{-i})} \cdot \frac{p(x_i | \bar{x}_{-i})}{p(x'_i | \bar{x}_{-i})} = 1$$

Slice sampling

$\bar{x} \sim p(\bar{x}) \Leftrightarrow (\bar{x}, u) \sim \text{Unif}($подграфик $p(\bar{x}))$

- $u \sim \text{Unif}(0, p(\bar{x}))$
- $\bar{x} \sim p(\bar{x} | u)$