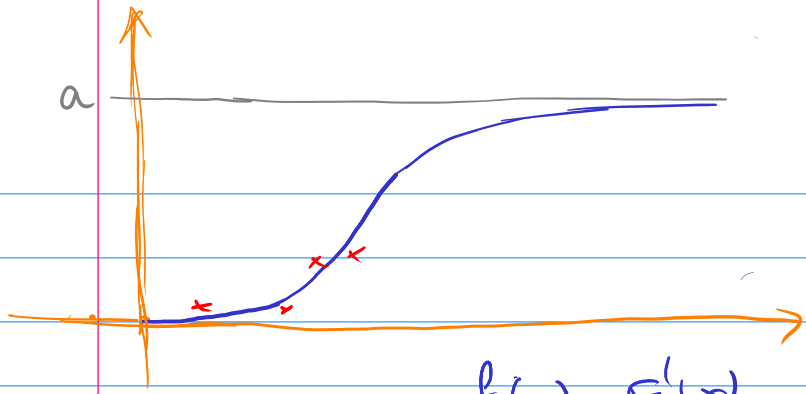



$$F(x) = \int_{-\infty}^{x} \mathcal{N}(y \mid \mu, \sigma^2)\, dy = \int_{-\infty}^{x} c \cdot e^{-\frac{1}{2\sigma^2}(y-\mu)^2} dy$$

$$f(x) = F'(x) = \underline{c e^{-\frac{1}{2\sigma^2}(x-\mu)^2}}$$

$$\ln f(x) = w_0 + w_1 x + w_2 x^2$$

---

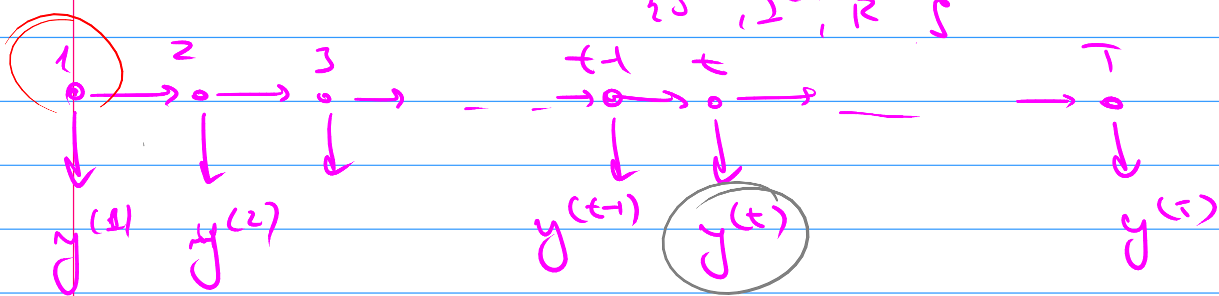
SIR – susceptible – infected – recovered

$X = \{x_1, \ldots, x_N\}$ $\qquad S \xrightarrow{} I \xrightarrow{} R$ $\qquad S^{(t)} = |\{i : x_i^{(t)} = S\}|$

$x_i^{(t)} \in \{S, I, R\}$ $\qquad I^{(t)}, \quad R^{(t)}$

$$\forall t \quad S^{(t)} + I^{(t)} + R^{(t)} = N$$

$\{S^{(t)}, I^{(t)}, R^{(t)}\}$




Параметры модели: $\qquad \bar{\theta} = \{\pi, \rho, \mu, \beta\}$

1) $p(x_i^{(1)} = I) = \pi \qquad p(x_i^{(1)} = S) = 1 - \pi$

2) $p(x_i^{(t)} \in y^{(t)} \mid x_i^{(t)} = I) = \rho$

$$p(y^{(t)} \mid I^{(t)}, \rho) = \text{Binomial}(y^{(t)} \mid I^{(t)}, \rho)$$

3) $S \rightarrow I \xrightarrow[\boxed{\mu}]{} R \qquad \boxed{R_0}$

$\underline{\beta = p(\text{заразился от одного контакта})}$

$$p(x_i^{(t)} = S \mid x_i^{(t-1)} = S) = (1-\beta)^{I^{(t-1)}}$$

$$p(x_i^{(t)} \mid x_i^{(t-1)}) = \begin{array}{c|ccc} & S & I & R \\ \hline S & (1-\beta)^{I^{(t-1)}} & 1-(1-\beta)^{I^{(t-1)}} & 0 \\ I & 0 & 1-\mu & \mu \\ R & 0 & 0 & 1 \end{array}$$

$$p(x, y \mid \theta) = \underline{p(x^{(1)} \mid \pi)}\, p(\underline{y^{(1)} \mid x^{(1)}})\, p(x^{(2)} \mid x^{(1)}) \ldots p(y^{(T)} \mid x^{(T)})$$

$$= \left[\pi^{I^{(1)}} (1-\pi)^{S^{(1)}}\right] \cdot \left[\prod_{t=1}^{T} \binom{I^{(t)}}{y^{(t)}} p^{y^{(t)}} (1-p)^{I^{(t)} - y^{(t)}}\right] \cdot$$

$$\cdot \left[\prod_{t=2}^{T} \prod_{j=1}^{N} p(x_j^{(t)} \mid \bar{x}_{-j}^{(t-1)}, \theta)\right]$$

$$p(\theta \mid y) \propto \cancel{p(\theta)}\, p(y \mid \theta) = \int p(x, y \mid \theta)\, dx$$

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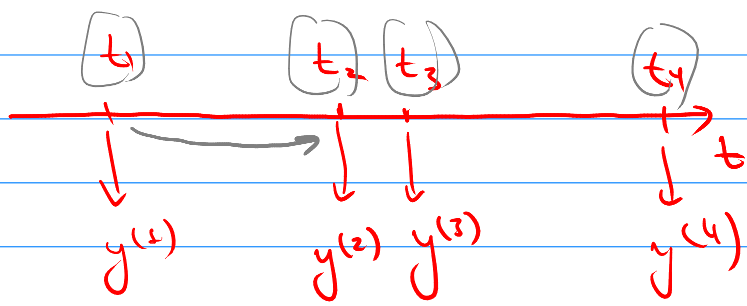
## Расширения SIR-моделей

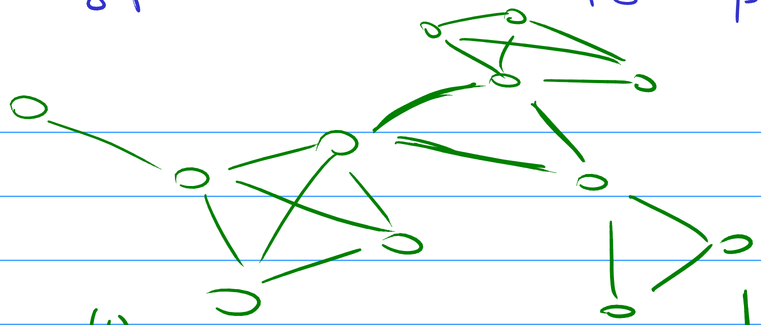
1) SIR ⇝ SEIR ⟶ SEIRS

exposed

2) Непрерывное время



3) Модель заражения $\qquad R_0 = \beta \cdot \underbrace{\mathbb{E}[\#\text{контактов}]}$



$$p(x_i^{(t)} = S \mid x_i^{(t-1)} = S) = (1-\beta)^{|I^{(t-1)} \cap Nei(x_i)|}$$

---

$$p(\theta|y) \propto p(\theta)p(y|\theta) = \int p(x,y|\theta)dx = \underbrace{\int p(x|\theta)\, p(y|x,\theta)\, dx}_{} =$$

$$= \mathbb{E}_x[p(y|x,\theta)] \approx \underbrace{\frac{1}{R}\sum_r} p(y|x_r,\theta)$$

$$\theta_0 \xrightarrow{\text{Gibbs}} X_0^{(2)}$$

MAP

$$\theta_1 \xrightarrow{\text{Gibbs}} X_1^{(2)}$$

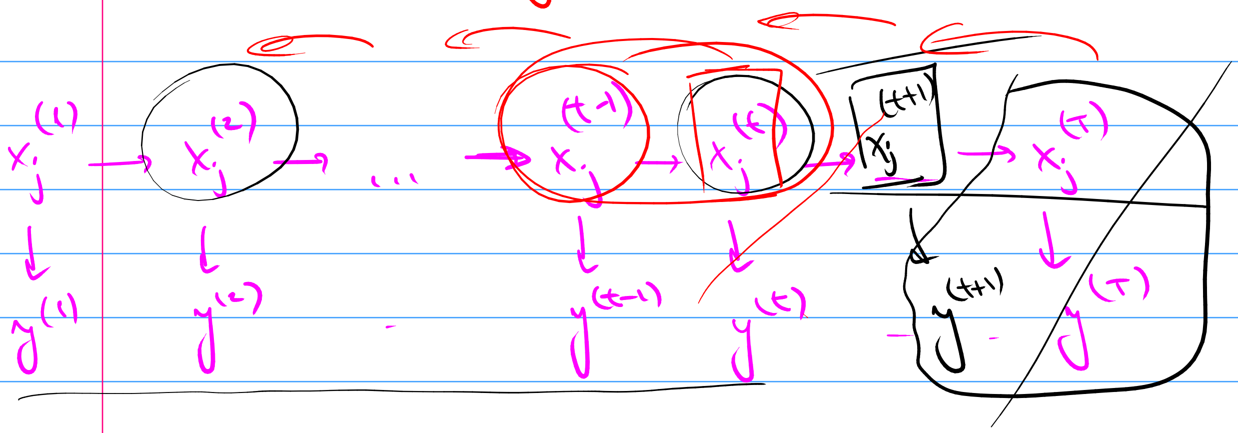
MAP

$$\theta_2 \xrightarrow{\text{Gibbs}} \ldots$$

$$\begin{array}{c|ccc} & I^{(t-1)} & & J^{(t-1)} \\ \hline S & (1-\beta) & 1-(1-\beta) & 0 \\ I & 0 & 1-\mu & \mu \\ R & 0 & 0 & 1 \\ \hline & S & I & R \end{array}$$

$j=1, \ldots, j=N$

$$\boxed{\bar{x}_j \sim p(\bar{x}_j \mid X_{-j}, y, \theta)}$$




Stochastic Viterbi algorithm

$$p(x_j^{(t)} \mid x_j^{(t+1)}, \ldots, x_j^{(T)}, \bar{y}, X_{-j}, \theta) =$$

$$= p(x_j^{(t)} \mid x_j^{(t+1)}, X_{-j}, y^{(1)}, \dots, y^{(t)}, \theta)$$

$$q_{j,s,s'}^{(t)} = p(x_j^{(t)}=s,\ x_j^{(t+1)}=s' \mid y^{(1)}, \dots, y^{(t)}, X_{-j}, \theta) \propto$$

$$\propto p(x_j^{(t)}=s,\ x_j^{(t+1)}=s',\ y^{(t)} \mid \bar{y}_{<t}, X_{-j}, \theta) =$$

$$= \underbrace{p(x_j^{(t)}=s \mid \bar{y}_{<t}, X_{-j}, \theta)}_{=\left(\sum_{s''} q_{j,s'',s}^{(t-1)}\right)} \cdot \underbrace{p(x_j^{(t+1)}=s' \mid x_j^{(t)}=s, \dots)\, p(y^{(t)} \mid x_j^{(t)}=s, \dots)}_{=p(x_j^{(t+1)}=s' \mid x_j^{(t)}=s, \theta)\cdot \mathrm{Binom}(y^{(t)} \mid I^{(t)}, p)} =$$

$$= \left(\sum_{s''} q_{j,s'',s}^{(t-1)}\right) \cdot p(x_j^{(t+1)}=s' \mid x_j^{(t)}=s, X_{-j}, \theta) \cdot \mathrm{Binom}(y^{(t)} \mid I_{-j}^{(t)} + [s=I], p)$$

## Gibbs sampling

1. $Q_j^{(t)} = \left( \; - \; q_{j,s,s'}^{(t)} \; - \; \right)$ — дин. прог.

2. Семпл. $x_j^{(T)}, x_j^{(T-1)}, \dots, x_j^{(1)}$:
   - $x_j^{(T)} \sim p(x_j^{(T)}=s \mid y, X_{-j}, \theta) = \sum_{s'} q_{j,s',s}^{(T)}$
   - $x_j^{(t)} \sim p(x_j^{(t)}=s \mid x_j^{(t+1)}=s', y^{(1)}, \dots, y^{(t)}, X_{-j}, \theta) \propto q_{j,s,s'}^{(t)}$

3. Выдаём $\bar{x}_j = x_j^{(1)}, \dots, x_j^{(T)}$

Алгоритм обучения:

1) Init $\theta, X$ так, чтобы было совм с $y$

2) for $j = 1 .. N, 1, .., N, 1, .., N, \_\_\_$ :

(i) $\bar{x}_j \sim p(\bar{x}_j | X_{-j}, Y, \theta)$ по стох. алг. Витерби

(ii) заменяю $\bar{x}_j$ в популяции

(iii) каждые $M$ шагов:

(М-шаг): обновить $\theta$ по $X$

М-шаг: $p(\theta | X, y) \propto \underbrace{p(\theta)}_{= p(\pi) p(\beta) p(\rho) p(\mu)} p(X, y | \theta)$

$p(\rho) = Beta(a_\rho, b_\rho)$

$$\log p(\theta | X, y) = const + (a_\pi - 1)\log \pi + (b_\pi - 1)\log(1 - \pi) +$$
$$+ \sum_{j=1}^{N} \left( [x_j^{(1)} = S] \log(1 - \pi) + [x_j^{(1)} = I] \log \pi \right) +$$

$$+ (a_\mu - 1)\log \mu + (b_\mu - 1)\log(1 - \mu) +$$
$$+ \sum_{t=1}^{T-1} \sum_{j=1}^{N} \left( [x_j^{(t)} = I, x_j^{(t+1)} = R] \cdot \log \mu + \right.$$
$$\left. + [x_j^{(t)} = I, x_j^{(t+1)} = I] \log(1 - \mu) \right) +$$

$$+ (a_\rho - 1)\log \rho + (b_\rho - 1)\log(1 - \rho) +$$
$$+ \sum_{t=1}^{T} \left( y^{(t)} \log \rho + (I^{(t)} - y^{(t)}) \log(1 - \rho) \right) +$$

$$P_j^{(t)} + N_j^{(t)} = I^{(t)}$$

$$+ (a_\beta - 1)\log \beta + (b_\beta - 1)\log(1 - \beta) +$$
$$+ \sum_{t=1}^{T-1} \sum_{j=1}^{N} [x_j^{(t)} = S] \cdot \left( P_j^{(t)} \log \beta + N_j^{(t)} \log(1 - \beta) \right)$$

$P_j^{(t)}$ — # конт., от к-рых $x_j$ заразился

$N_j^{(t)}$ — —"— не заразился

$$P_j^{(t)} + N_j^{(t)} = I^{(t)}$$

- если $x_j^{(t+1)} = S$, то $P_j^{(t)} = 0$, $N_j^{(t)} = I^{(t)}$
- если $x_j^{(t+1)} = I$, то $P_j^{(t)} \geq 1$, $N_j^{(t)} = I^{(t)} - P_j^{(t)}$

$$E\left[P_j^{(t)} \middle| x_j \text{ зараж.}\right] = I^{(t)} \cdot p(\text{зараж. при 1 конт.} \mid \text{заразился}) = I^{(t)} \cdot \frac{\beta}{1-(1-\beta)^{I^{(t)}}}$$

$$a_\pi' = a_\pi + \sum_{j=1}^{N} [x_j^{(1)} = I] \qquad b_\pi' = b_\pi + \sum_{j=1}^{N} [x_j^{(1)} = S]$$

$$a_\mu' = a_\mu + \sum_{t=1}^{T-1} \sum_{j=1}^{N} [x_j^{(t)} = I, x_j^{(t+1)} = R] \qquad b_\mu' = b_\mu + \sum_{t=1}^{T-1} \sum_{j=1}^{N} [x_j^{(t)} = I, x_j^{(t+1)} = I]$$

$$a_\rho' = a_\rho + \sum_{t=1}^{T} y^{(t)} \qquad b_\rho' = b_\rho + \sum_{t=1}^{T} \left(I^{(t)} - y^{(t)}\right)$$

$$a_\beta' = a_\beta + \sum_{t=1}^{T-1} \sum_{j=1}^{N} [x_j^{(t)} = S, x_j^{(t+1)} = I] \cdot I^{(t)} \cdot \frac{\beta}{1-(1-\beta)^{I^{(t)}}}$$

$$b_\beta' = b_\beta + \sum_{t=1}^{T-1} \sum_{j=1}^{N} \left( [x_j^{(t)} = S, x_j^{(t+1)} = S] \cdot I^{(t)} + [x_j^{(t)} = S, x_j^{(t+1)} = I] \cdot I^{(t)} \cdot \left(1 - \frac{\beta}{1-(1-\beta)^{I^{(t)}}}\right) \right)$$