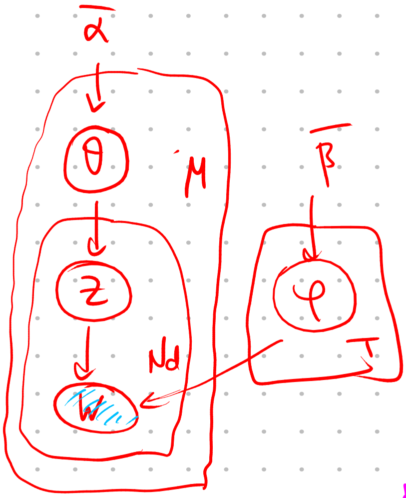
LDA



$$p(W, Z, \Theta, \Phi) = \left(\prod_{t=1}^{T} p(\bar\varphi|\bar\beta)\right) \times$$

$$\times \prod_{j=1}^{M}\left(p(\bar\theta_j|\bar\alpha)\cdot \prod_{n=1}^{N_j} p(z_{jn}|\bar\theta_j)\, p(w_{jn}|z_{jn}, \Phi)\right)$$

$$p(\bar\theta_j|\bar\alpha) = \frac{1}{B(\bar\alpha)} \prod_{t=1}^{T} \theta_{jt}^{\alpha_t - 1}$$

$$p(\bar\varphi_t|\bar\beta) = \frac{1}{B(\bar\beta)} \prod_{v=1}^{V} \varphi_{tv}^{\beta_v - 1}$$

$$p(\Theta, \Phi \mid W, \bar\alpha, \bar\beta) \longrightarrow \max$$

$$\propto p(\Theta, \Phi, W \mid \bar\alpha, \bar\beta) = \sum_Z p(\Theta, \Phi, W, Z \mid \bar\alpha, \bar\beta)$$

$$q(Z, \Theta, \Phi) \approx p(Z, \Theta, \Phi \mid W, \bar\alpha, \bar\beta)$$

$$\log p(W|\bar\alpha, \bar\beta) = \log \iint_{\Phi\,\Theta} \sum_Z p(Z, \Theta, \Phi, W|\bar\alpha, \bar\beta) \cdot \frac{q(Z, \Theta, \Phi)}{q(Z, \Theta, \Phi)} \geq$$

$$\geq \iint_{\Phi\,\Theta} \sum_Z q(Z, \Theta, \Phi) \log \frac{p(Z, \Theta, \Phi, W|\bar\alpha, \bar\beta)}{q(Z, \Theta, \Phi)}\, d\Theta\, d\Phi =$$

$$= \underbrace{-\iint \sum q(Z, \Theta, \Phi) \log \frac{q(Z, \Theta, \Phi)}{p(Z, \Theta, \Phi|W, \bar\alpha, \bar\beta)}\, d\Theta\, d\Phi}_{-KL(q\|p(Z, \Theta, \Phi|W))} + \log p(W|\bar\alpha, \bar\beta)$$

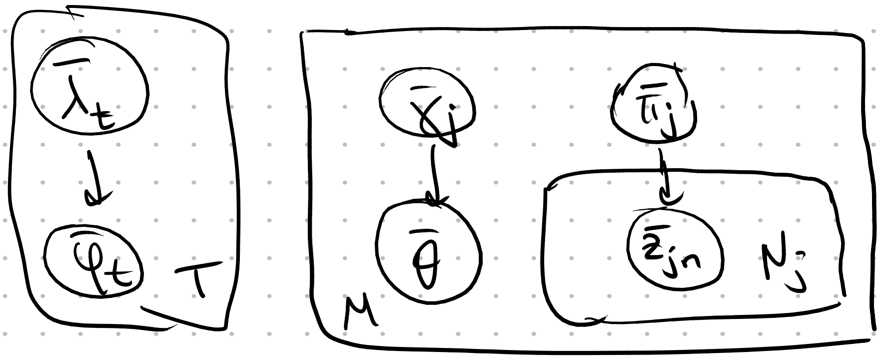
$$L(q) = \iint \sum q(Z, \Theta, \Phi) \log \frac{p(Z, \Theta, \Phi, W|\bar\alpha, \bar\beta)}{q(Z, \Theta, \Phi)}\, d\Theta\, d\Phi \longrightarrow \max$$

$$= E_q\left[\log p(Z, \Theta, \Phi, W|\bar\alpha, \bar\beta)\right] \xrightarrow[q]{} \max$$

$$q(Z,\Theta,\Phi) = q(Z|\bar{\pi})\cdot q(\Phi,\Theta|\bar{\lambda},\bar{\gamma})$$

$$L(q) = E_q\Big[\sum_{t=1}^{T}\log p(\bar{\varphi}|\bar{\beta}) + \sum_{j=1}^{M}\log p(\bar{\theta}_j|\bar{\alpha}) + \sum_{j=1}^{M}\sum_{n=1}^{N_j}\big(\log p(z_{jn}|\bar{\theta}_j) + \log p(w_{jn}|z_{jn},\bar{\Phi})\big)\Big] \xrightarrow[\Lambda,\Gamma,\Pi]{} \max$$

$$q(Z,\Theta,\Phi|\Gamma,\Lambda,\Pi) = \left(\prod_{t=1}^{T} q_t(\bar{\varphi}_t|\bar{\lambda}_t)\right)\cdot\prod_{j=1}^{M}\left(q_j(\bar{\theta}_j|\bar{\gamma}_j)\prod_{n=1}^{N_j} q_j(\bar{z}_{jn}|\bar{\pi}_j)\right)$$



$$q_t(\bar{\varphi}_t|\bar{\lambda}_t) = Dir(\bar{\varphi}_t|\bar{\lambda}_t)$$
$$q_j(\bar{\theta}_j|\bar{\gamma}_j) = Dir(\bar{\theta}_j|\bar{\gamma}_j)$$
$$q_j(z_{jn}|\bar{\pi}_j) = Mult(z_{jn}|\bar{\pi}_j)$$

1) Φ – „гиперпараметр", $\underline{\Phi \text{ fixed}}$

$$q(Z,\Theta|\Gamma,\Lambda) \approx p(Z,\Theta|W,\Phi,\bar{\alpha},\bar{\beta})$$

$$KL(q_j\|p_j) = \int_{\bar{\theta}_j}\sum_{\bar{z}_j} q(\bar{z}_j,\bar{\theta}_j|\bar{\gamma}_j,\bar{\pi}_j)\cdot\log\frac{q(\bar{z}_j,\bar{\theta}_j|\bar{\gamma}_j,\bar{\pi}_j)}{p(\bar{z}_j,\bar{\theta}_j|\bar{w}_j,\Phi,\bar{\alpha},\bar{\beta})}\,d\bar{\theta}_j \xrightarrow[\gamma,\bar{\pi}]{} \min$$

$$L(\bar{\gamma}_j,\bar{\pi}_j) \xrightarrow[\gamma,\bar{\pi}]{} \max$$

$$L(\bar{\gamma}_j,\bar{\pi}_j) = \int_{\bar{\theta}_j}\sum_{\bar{z}_j} q(\_)\cdot\log\frac{p(\bar{w}_j,\bar{z}_j,\bar{\theta}_j|\Phi,\bar{\alpha},\bar{\beta})}{q(\_\,\_|\_)}\,d\bar{\theta}_j =$$

$$= E_q\left[\log p(\bar{w}_j,\bar{z}_j,\bar{\theta}_j|\Phi,\bar{\alpha},\bar{\beta})\right] - E_q\left[\log q(\bar{z}_j,\bar{\theta}_j|\bar{\gamma}_j,\bar{\pi}_j)\right]$$

$$= \underline{E_q\left[\log p(\bar{\theta}_j|\bar{\alpha})\right]} + \underline{E_q\left[\log p(\bar{z}_j|\bar{\theta}_j)\right]} + \underline{E_q\left[\log p(\bar{w}_j|\bar{z}_j,\Phi)\right]} -$$

$$- \mathbb{E}_q[\log q(\bar{\theta}_j | \bar{\gamma}_j)] - \mathbb{E}_q[\log q(\bar{z}_j | \bar{\pi}_j)]$$

① $$\mathbb{E}_q[\log p(\bar{\theta}_j | \bar{\alpha})] = \mathbb{E}_q\left[-\log B(\bar{\alpha}) + \sum_{t=1}^{T} (\alpha_t - 1) \log \theta_{jt}\right] =$$

$$\log \Gamma(\sum \alpha_t) - \sum \log \Gamma(\alpha_t) + \sum_t (\alpha_t - 1) \mathbb{E}_q[\log \theta_{jt}]$$

$$B(\bar{\alpha}) = \frac{\prod \Gamma(\alpha_t)}{\Gamma(\sum \alpha_t)}$$

$$p(\bar{x}) = h(\bar{x}) g(\bar{\eta}) \cdot e^{\bar{\eta}^T \bar{u}(\bar{x})} = h(x) e^{\bar{\eta}^T \bar{u}(\bar{x}) - \log g(\bar{\eta})}$$

$$\nabla_{\bar{\eta}} \int h(x) e^{\bar{\eta}^T \bar{u}(\bar{x}) - \log g(\bar{\eta})} d\bar{x} = \nabla_{\bar{\eta}} 1 = 0$$

$$\Downarrow$$

$$\int h(\bar{x}) \left(\bar{u}(\bar{x}) - \nabla_{\bar{\eta}} \log g(\bar{\eta}) \right) \cdot e^{\bar{\eta}^T \bar{u}(\bar{x}) - \log g(\bar{\eta})} d\bar{x} = 0$$

$$\int \bar{u}(\bar{x}) \underbrace{h(\bar{x}) e^{\cdots}}_{= p(\bar{x})} d\bar{x} - \nabla_\eta \log g(\bar{\eta}) \cdot \cancel{\int \underbrace{h(x) e}_{p(\bar{x})} d\bar{x}} = 0$$

$$\boxed{\mathbb{E}_{p(\bar{x})}[\bar{u}(\bar{x})] = \nabla_\eta \log g(\bar{\eta})}$$

$$\bar{u}(\bar{x}) = \log \bar{x}$$
$$g(\bar{\eta}) = \frac{1}{B(\bar{\alpha})}$$
$$\bar{\eta} = \bar{\alpha} - 1$$

$$\mathrm{Dir}(\bar{x} | \bar{\alpha}) = \frac{1}{B(\bar{\alpha})} \prod x_i^{\alpha_i - 1} = \frac{1}{B(\bar{\alpha})} e^{(\bar{\alpha}-1)^T \log \bar{x}}$$

$$\mathbb{E}_p[\log \bar{x}] = -\nabla_{\bar{\alpha}} \log B(\bar{\alpha})$$

$$\Psi(x) = \frac{d \log \Gamma(x)}{dx}$$

$$\boxed{E_{p(\bar{x})}[\log x_i] = -\frac{\partial \log B(\bar{\alpha})}{\partial x_i} = \psi(\alpha_i) - \psi(\textstyle\sum \alpha_j)}$$

$$E_q[\log \theta_{jt}] = \psi(\gamma_{jt}) - \psi\left(\sum_{s=1}^{T} \gamma_{js}\right)$$

$$E_q[\log p(\bar{\theta}_j|\bar{\alpha})] = \log\Gamma\left(\sum_t \alpha_t\right) - \sum_t \log\Gamma(\alpha_t) + \sum_t (\alpha_t - 1)\left(\psi(\gamma_{jt}) - \psi\left(\sum_s \gamma_{js}\right)\right)$$

(1)

(2) $$E_q[\log p(\bar{z}_j|\bar{\theta}_j)] = E_q\left[\sum_{n=1}^{N_j} \log \underbrace{p(z_{jn}|\bar{\theta}_j)}_{=\prod_t \theta_{jt}^{[z_{jn}=t]}}\right] =$$

$$= E_q\left[\sum_{n=1}^{N_j}\sum_{t=1}^{T} [z_{jn}=t] \cdot \log \underbrace{\theta_{jt}}_{\psi(.)-\psi(.)}\right] =$$

$$= \sum_n \sum_t \underbrace{E_q[[z_{jn}=t]]}_{=\pi_{jnt}} \cdot E_q[\log \theta_{jt}]$$

$$E_q[\log p(\bar{z}_j|\bar{\theta}_j)] = \sum_{n=1}^{N_j}\sum_{t=1}^{T} \pi_{jnt}\left(\psi(\gamma_{jt}) - \psi\left(\sum_s \gamma_{js}\right)\right)$$

(2)

(3) $$E_q[\log p(\bar{w}_j|\bar{z}_j, \Phi)] = E_q\left[\sum_n \sum_{t=1}^{T}\sum_{v=1}^{V} [z_{jn}=t][w_{jn}=v] \log \varphi_{tv}\right]$$

$$= \sum_{n=1}^{N_j}\sum_{t=1}^{T}\sum_{v=1}^{V} \left([w_{jn}=v] \cdot \pi_{jnt} \cdot \log \varphi_{tv}\right)$$

(4) $$E_q[\log q(\bar{\theta}_j|\bar{\gamma}_j)] = \log\Gamma\left(\sum_t \gamma_{jt}\right) - \sum_t \log\Gamma(\gamma_{jt}) + \sum_t (\gamma_{jt} - 1)\left(\psi(\gamma_{jt}) - \psi\left(\sum_s \gamma_{js}\right)\right)$$

$$⑤\ \mathbb{E}_q\left[\log q(\bar{z}_j \mid \bar{\pi}_j)\right] = \mathbb{E}_q\left[\sum_{n=1}^{N_j} \sum_t [z_{jn}=t] \log \pi_{jnt}\right] =$$

$$= \sum_n \sum_t \pi_{jnt} \log \pi_{jnt}$$

$$L(\bar{\gamma}_j, \bar{\pi}_j) = \underset{\gamma}{①} + \underset{\pi,\gamma}{②} + \underset{\pi}{③} - \underset{\gamma}{④} - \underset{\pi}{⑤} \xrightarrow[\bar{\gamma}_j,\ \bar{\pi}_j]{} \max$$

при усл. $\forall n \ \sum_t \pi_{jnt} = 1$, $\pi_{jnt} \geq 0$

$$\bar{\pi}_j:\ L(\bar{\pi}_j) = \sum_{n,t} \pi_{jnt}\Big(\psi(\gamma_{jt}) - \psi\big(\sum_s \gamma_{js}\big) + \sum_\delta [w_{jn}=\delta] \log \varphi_{t\delta} - \log \pi_{jnt}\Big) + \sum_{n=1}^{N_j} \lambda_n\Big(\sum_s \pi_{jns} - 1\Big)$$

$$\frac{\partial L}{\partial \pi_{jnt}} = \psi(\gamma_{jt}) - \psi\big(\sum_s \gamma_{js}\big) + \log \varphi_{t,w_{jn}} - \log \pi_{jnt} - 1 + \lambda_n = 0$$

$$\log \pi_{jnt} = (\lambda_n - 1) + \psi(\_) - \psi(\_) + \log \varphi_\_$$

$$\pi_{jnt} = e^{\lambda_n - 1} \cdot e^{—\ —}$$

$$\boxed{\pi^*_{jnt} \propto e^{\psi(\gamma_{jt}) - \psi(\sum_s \gamma_{js}) + \log \varphi_{t,w_{jn}}}}$$

$$\bar{\gamma}_j:\ L(\bar{\gamma}_j) = \log \Gamma\big(\sum_t \alpha_t\big) - \sum_t \log \Gamma(\alpha_t) + \sum_t (\alpha_t - 1)\Big(\psi(\gamma_{jt}) - \psi\big(\sum_s \gamma_{js}\big)\Big)$$

$$+ \sum_n \sum_t \pi_{jnt}\Big(\psi(\gamma_{jt}) - \psi\big(\sum_s \gamma_{js}\big)\Big) - \log \Gamma\big(\sum_t \gamma_{jt}\big) + \sum_t \log \Gamma(\gamma_{jt}) -$$

$$- \sum_t (\gamma_{jt} - 1)\Big(\psi(\gamma_{jt}) - \psi\big(\sum_s \gamma_{js}\big)\Big) =$$

$$= \sum_{t} \left(\psi(\gamma_{jt}) - \psi\left(\sum_s \gamma_{js}\right) \right) \left(\alpha_t - \cancel{1} + \sum_n \pi_{jnt} - \gamma_{jt} \cancel{+1} \right) + \qquad \gamma_j \to \max$$

$$+ \underbrace{\log\Gamma\left(\sum \alpha_t\right) - \sum_t \log\Gamma(\alpha_t)}_{\text{Const}} - \log\Gamma\left(\sum_t \gamma_{jt}\right) + \sum_t \log\Gamma(\gamma_{jt})$$

$$\frac{\partial L}{\partial \gamma_{jt}} = -\psi'\left(\sum_s \gamma_{js}\right) \cdot \sum_{s=1}^{T} \left(\alpha_s + \sum_n \pi_{jns} - \gamma_{js} \right) -$$

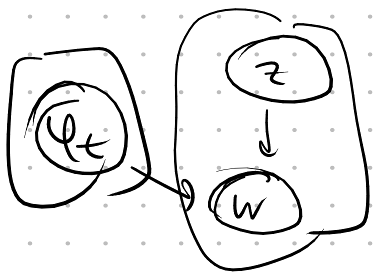
$$- \cancel{\psi(\gamma_{jt})} + \cancel{\psi\left(\sum_s \gamma_{js}\right)} + \psi'(\gamma_{jt}) \left(\alpha_t + \sum_n \pi_{jnt} - \gamma_{jt} \right) -$$

$$- \cancel{\psi\left(\sum_s \gamma_{js}\right)} + \cancel{\psi(\gamma_{jt})}$$

$$\forall t: \ \psi'(\gamma_{jt}) \left(\alpha_t + \sum_n \pi_{jnt} - \gamma_{jt} \right) - \psi'\left(\sum_s \gamma_{js}\right) \sum_s \left(\alpha_s + \sum_n \pi_{jns} - \gamma_{js} \right) = 0$$

$$\boxed{\gamma_{jt}^{*} = \alpha_t + \sum_n \pi_{jnt}}$$

$\boxed{\text{I}}$



EM-algorithm w.r.t. Φ

- E-step: $KL(q_j \| p_j) \to \min \quad \forall j$

  находим Π, Γ

- M-step: $\varphi_{tv} \propto \sum_{j=1}^{M} \sum_{n=1}^{N_j} [w_{jn} = v] \cdot \pi_{jnt}$

$\boxed{\text{II}}$ $q(Z, \Theta, \Phi \mid \Gamma, \Pi, \Lambda) \approx p(Z, \Theta, \Phi \mid W, \alpha, \beta)$

$$L(\_) = \sum_{j=1}^{M} L(\pi_j, \gamma_j) + \sum_t \mathbb{E}_q\left[\log p(\varphi_t \mid \beta)\right] - \sum_t \mathbb{E}_q\left[\log q(\varphi_t \mid \lambda_t)\right]$$

$$\boxed{\lambda_{tv}^{*} = \beta_v + \sum_{j=1}^{M} \sum_{n=1}^{N_j} [w_{jn} = v] \, \pi_{jnt}}$$