model parameters   data

prior   likelihood θ_{ML}

$$\theta_{MAP} \quad p(\theta|D) = \frac{p(\theta)\,p(D|\theta)}{p(D)} \xrightarrow{} \max$$

posterior   evidence

max

$$p(\theta|D) \propto p(\theta)\,p(D|\theta)$$

Монетка $\theta = p(\text{орёл})$

$D = hhttthh$

$$p(D|\theta) = \theta^4(1-\theta)^3 \xrightarrow[\theta]{} \max$$

$$\theta^n(1-\theta)^m \to \max$$

$$\theta_{ML} = \operatorname{argmax}\, p(D|\theta)$$

$$\theta_{ML} = \frac{n}{n+m}$$

$$\frac{\partial p(D|\theta)}{\partial \theta} = 4\theta^3(1-\theta)^3 - 3\theta^4(1-\theta)^2 = 0$$

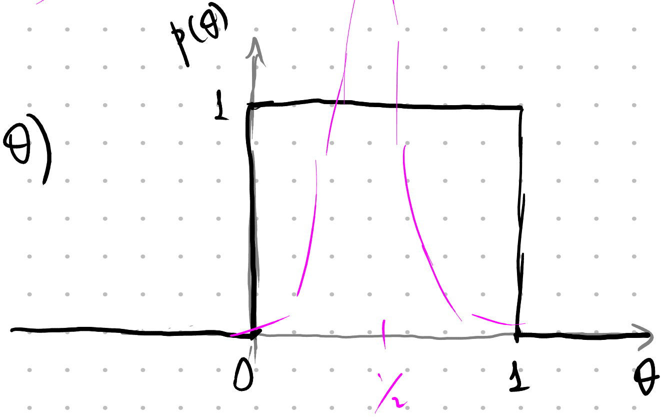
$$\theta^3(1-\theta)^2\big(4(1-\theta)-3\theta\big) = 0$$

$$\theta = 0,\ 1,\ 4/7$$

$D = t$

$$p(D|\theta) = 1-\theta \qquad \theta_{ML} = 0$$

$p(\theta)$



$$p(\theta) = \begin{cases} 1, & \theta \in [0,1] \\ 0, & \theta \notin [0,1] \end{cases}$$

$$p(\theta|D) \propto p(\theta)p(D|\theta) = \begin{cases} \theta^n(1-\theta)^m, & \theta \in [0,1] \\ 0, & \theta \notin [0,1] \end{cases}$$

$$p(D) = \int p(\theta)p(D|\theta)\,d\theta = \int_0^1 \theta^n(1-\theta)^m\,d\theta = B(n+1, m+1)$$

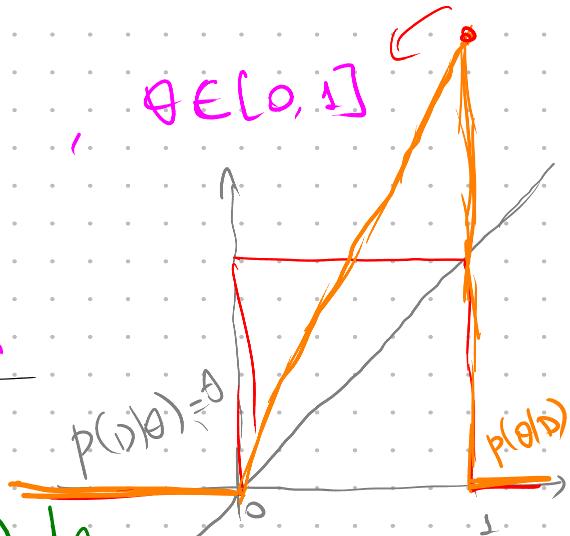
$$B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$$

$$= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \qquad \Gamma(n) = (n-1)!$$

$$B(n+1, m+1) = \frac{\Gamma(n+1)\Gamma(m+1)}{\Gamma(n+m+2)} =$$

$$p(D) = \frac{n!\,m!}{(n+m+1)!}$$

$$p(\theta|D) = \frac{(n+m+1)!}{n!\,m!}\cdot\theta^n(1-\theta)^m, \quad \theta\in[0,1]$$



$$\theta_{MAP} = \arg\max_\theta p(\theta|D) = \frac{n}{n+m}$$

$$p(\text{heads}\,|\,D) = \int_{-\infty}^{\infty} p(\text{heads}, \theta\,|\,D)\,d\theta = E_{p(\theta|D)}[\theta]$$

$$= \int \underbrace{p(\text{heads}\,|\,\theta, \cancel{D})}_{\theta} \cdot \underbrace{p(\theta|D)}_{\text{posterior}}\,d\theta =$$

$$= \int_0^1 \theta\cdot\frac{(n+m+1)!}{n!\,m!}\,\theta^n(1-\theta)^m\,d\theta = \frac{(n+m+1)!}{n!\,m!}\cdot\frac{(n+1)!\,m!}{(n+m+2)!}$$

predictive distribution

$$\boxed{p(\text{heads}\,|\,D) = \frac{n+1}{n+m+2}}$$

Laplace's Rule of succession