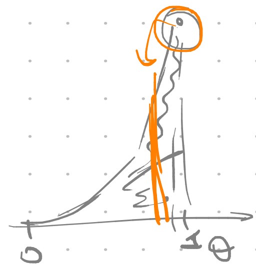
① Hot hand fallacy

② Оценка размер. распред.

$1, \ldots, N$ Unif



1 танк: $m = D$,

$$p(m \mid N) = \begin{cases} 1/N, & N \geq m \\ 0, & N < m \end{cases}$$

$$N_{ML} = m$$

$D = \{m_1, m_2, \ldots, m_k\}$ $m = \max_k m_k$

$$E[m \mid N, k] \qquad m = f(N, k)$$

1 танк: $E[m \mid N] = E[m_1 \mid N] = \frac{N+1}{2}$

$$\Rightarrow \boxed{\hat{N} = 2m - 1}$$

$k > 1$:

$$p(m \mid N, k) = \frac{\#\{\text{благ. } k \text{ чисел от 1 до } N \text{ с макс. } m\}}{\#\{\text{благ. } k \text{ ч. от 1 до } N\}} = \frac{\binom{m-1}{k-1}}{\binom{N}{k}}$$

$$\boxed{E[m \mid N, k]} = \sum_{m=k}^{N} m \cdot \frac{\binom{m-1}{k-1}}{\binom{N}{k}} = \binom{N}{k}^{-1} \cdot \sum_{m=k}^{N} \frac{k \cdot m \cdot (m-1)!}{k \cdot (k-1)!\,(m-k)!} =$$

$$= k \cdot \binom{N}{k}^{-1} \cdot \underbrace{\sum_{m=k}^{N} \binom{m}{k}} = k \cdot \binom{N}{k}^{-1} \cdot \binom{N+1}{k+1} =$$

$$\boxed{\sum_{a=c}^{b} \binom{a}{c} = \binom{b+1}{c+1}}$$

$$= k \cdot \frac{(N+1)!}{(k+1)!\,\cancel{(N-k)!}} \cdot \frac{k!\,\cancel{(N-k)!}}{N!} =$$

$$= \frac{k}{k+1}(N+1) \quad = m$$

$$\boxed{\hat{N} = \frac{k+1}{k} \cdot m - 1 = m + \left(\frac{m}{k} - 1\right)}$$

$$p(N \mid m, k) = \frac{p(m \mid N, k) \cdot p(N \mid k)}{p(m \mid k)} \qquad m = \max m_i$$

$$p(m \mid N, k) = \binom{m-1}{k-1} \Big/ \binom{N}{k}$$

$$p(m \mid k) = \sum_{N=m}^{\infty} p(m \mid N, k)\, p(N \mid k)$$

$$p(N \mid k) = \mathrm{Unif}(k, k+1, \dots, \Omega)$$

$$p(N \mid k) = \frac{1}{\Omega - k + 1} \quad \text{на} \quad k \le N \le \Omega$$

$$p(N \mid m, k) = \frac{\left(\cancel{\binom{m-1}{k-1}} \Big/ \binom{N}{k}\right) \cdot \cancel{\frac{1}{\Omega - k + 1}}}{\sum_{N'=m}^{\Omega} \cancel{\binom{m-1}{k-1}} \Big/ \binom{N'}{k} \cdot \cancel{\frac{1}{\Omega - k + 1}}} =$$

$$= \frac{\binom{N}{k}^{-1}}{\sum_{N'=m}^{\Omega} \binom{N'}{k}^{-1}}$$

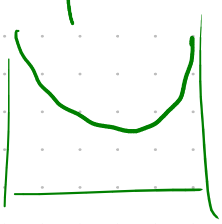
$$k \ge 2 \Rightarrow \sum_{N'=m}^{\infty} \binom{N'}{k}^{-1} < \infty$$

$$\boxed{p(N \mid k) \text{ "=" } \mathrm{Unif}(k, k+1, \dots, \infty)}$$

improper prior

$$p(\theta)^{(\alpha,\beta)} = \mathrm{Beta}(\alpha, \beta) \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}$$

$$p(\theta \mid 0, 0) \propto \frac{1}{\theta(1-\theta)} \quad \text{– improper prior}$$



$$p(a \mid m, k) \propto \frac{1}{\binom{a-m}{k}}, \quad a \ge m, \dots$$

$$p(N \mid m, k) = \frac{\binom{N}{k}^{-1}}{\sum_{N'=m}^{\infty} \binom{N'}{k}^{-1}} = \binom{m}{k}^{-1} \cdot \frac{m}{k-1}$$

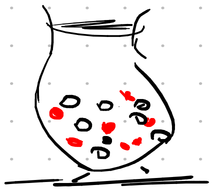
$$= \frac{k-1}{m} \cdot \frac{\binom{m}{k}}{\binom{N}{k}},$$ для $N = m, m+1, \ldots$

(если $k \geq 2$)

$$E[N|m,k] = \frac{k-1}{k-2}(m-1), \quad k>2$$

$$Var[N|m,k] = \frac{(k-1)(m-1)(m-k+1)}{(k-2)^2(k-3)}, \quad k>3$$

③ Пример Джейнса / Jaynes



N шаров
R красных
N−R белых

$D = (n, r)$

$$p(r|N,R,n) = \frac{\binom{R}{r}\binom{N-R}{n-r}}{\binom{N}{n}}$$ — hypergeometric distr.

$$\frac{N-R-(n-r)+1}{N-r-(n-r)+1}$$

$$p(\underbrace{Red \ldots Red}_{r}\ \underbrace{White \ldots White}_{n-r}) = \left(\frac{R}{N}\cdot\frac{R-1}{N-1}\cdots\frac{R-r+1}{N-r+1}\right)\left(\frac{N-R}{N-r}\cdot\frac{N-R-1}{N-r-1}\cdots\right)$$

$$\times \binom{n}{r} \quad \frac{n!}{r!(n-r)!}\cdot\frac{R!}{(R-r)!}\cdot\frac{\cancel{(N-r)!}}{N!}\cdot\frac{(N-R)!}{(N-R-(n-r))!}\cdot\frac{(N-n)!}{\cancel{(N-r)!}}$$

$$p(N,R|D) = \frac{p(D|N,R)\,p(N,R)}{p(D)} = \frac{p(D|N,R)\,p(R|N)\,p(N)}{\sum_{N=0}^{\infty}\sum_{R=0}^{N} p(D|N,R)\,p(R|N)\,p(N)}$$

$$p(N) \text{ - prior} \xrightarrow{D=(n,r)} p(N|D) \propto \begin{cases} p(N), & N \geq n \\ 0, & \text{— " —} \end{cases}$$

posterior

$$p(N|D) = \frac{p(N)\,p(D|N)}{p(D)} = \frac{p(N)\cdot \sum_{R=0}^{N} p(D|N,R)\,p(R|N)}{p(D)}$$

prior

$$\sum_{R=0}^{N} p(D|N,R)\,p(R|N) = \begin{cases} f(n,r), & N \geq n \\ 0, & N < n \end{cases}$$

$p(D|N,R) = p(r|N,R,n)\,p(n|N,R)$

$$\boxed{\sum_{R=0}^{N} \binom{R}{r}\binom{N-R}{n-r} p(R|N) = f(n,r)\cdot\binom{N}{n}}$$

1) $\text{Unif}(0,1,\dots,N)$ $\qquad p(R|N) = \begin{cases} \frac{1}{N+1}, & 0,\dots,N \\ 0 & \end{cases}$

$$\frac{1}{N+1}\sum_{R=0}^{N}\binom{R}{r}\binom{N-R}{n-r} = \frac{1}{N+1}\cdot\binom{N+1}{n+1} = \frac{1}{\cancel{N+1}}\cdot\frac{\cancel{(N+1)}\,N!}{(n+1)\,n!\,(N-n)!}$$

$0 \dots R \dots N$

$N+1$ $\quad (n+1)$ $\quad r$ $\quad n-r$

$= f(n,r)$

$$\binom{N}{k} = \frac{N!}{k!(N-k)!}$$

$$p(R|D,N) = \frac{\frac{1}{\cancel{N+1}}\cdot\binom{R}{r}\binom{N-R}{n-r}}{\cancel{\binom{N}{n}}} \Big/ \frac{\sum_{R=0}^{N}\frac{1}{\cancel{N+1}}\binom{R}{r}\binom{N-R}{n-r}}{\cancel{\binom{N}{n}}} =$$

(n,r)

$= \binom{N+1}{n+1}$

$$= \frac{1}{\binom{N+1}{n+1}}\cdot\binom{R}{r}\binom{N-R}{n-r} \quad \leftarrow \text{posterior}$$

$$n=r=1 \implies p(R|D,N) = \frac{2R}{N(N+1)}$$

$$R_{MAP} = \left\lfloor \frac{r}{n}(N+1)\right\rfloor$$

$$E[R \mid D, N] = \frac{(N+2)(r+1)}{n+2} - 1$$

$$\frac{E[R] - r}{N-n} = \frac{(N+2)(r+1) - (n+2)r - (n+2)}{(N-n)(n+2)} =$$

$$= \frac{Nr + \cancel{2r} + N + \cancel{2} - nr - \cancel{2r} - n - \cancel{2}}{(\ldots)} = \frac{(N-n)(r+1)}{(N-n)(n+2)}$$

$$= \frac{r+1}{n+2}$$

правило Лапласа

2) „Знаем", что в урне есть кр. и есть белый шар.

$R \geq 1$

$N - R \geq 1$

$$p(R \mid N) = \begin{cases} \frac{1}{N-1}, & R = 1, \ldots, N-1 \\ 0, & \text{—\textquotedbl—} \end{cases}$$

$$\sum_{R=1}^{N-1} \boxed{\binom{R}{r}} \binom{N-R}{n-r} = \binom{N+1}{n+1} - [r=0] \cdot \binom{N}{n} - [r=n] \cdot \binom{N}{n}$$

$$p(R \mid D, N) = \frac{\binom{R}{r}\binom{N-R}{n-r}}{\binom{N+1}{n+1} - \left([r=0] + [r=n]\right) \cdot \binom{N}{n}}$$

$r = 0 \Rightarrow$

$$p(R \mid D = (0, n), N) = \binom{N-R}{n} \Big/ \underbrace{\binom{N+1}{n+1} - \binom{N}{n}}_{= \binom{N}{n+1}}$$