Пример Даламбера

① $R \sim \cancel{Unif[0, N]}$ ② $R \sim Unif([1, N-1])$

$D = (n, r) = (2, 1)$

③ Uninformative prior

$p(R) = ?$ $\times$ $p(r=1 \mid N, R, n=2) = Unif$?

$$= \frac{\binom{R}{2}\binom{N-R}{n-2}}{\binom{N}{n}} = \frac{2R(N-R)}{N(N-1)}$$

$$p(R) \propto \frac{1}{R(N-R)}$$

uninformative prior

$$\boxed{p(R) = \frac{Const}{R(N-R)}, \quad R = 1, \ldots, N-1}$$

$$\frac{(N-2)!}{(n-2)!\,(N-R-(n-2))!}$$

$D = (2,1)$

$$p(R \mid D = (n, r), N) = \frac{Const}{R(N-R)} \cdot \binom{R}{r}\binom{N-R}{n-r} =$$

$$= \frac{Const}{r(n-r)} \cdot \binom{R-1}{r-1}\underbrace{\binom{N-R-1}{n-r-1}}_{r=1}$$

$$\sum_{a=0}^{b=N} \binom{a}{x}\binom{b-a}{y-x} = \binom{b+1}{y+1}$$

$a \le R$, $x \le r$, $b = N$, $y = n$

$$\sum_{R'=0}^{N-1} \binom{R'}{\underbrace{r-1}_{x}}\binom{N-2-R'}{\underbrace{n-2}_{y}-\underbrace{(r-1)}_{x}} = \binom{N}{n-1}$$

④ Binomial monkey prior

$g = pr[\text{красный}]$

$E[R] = g \cdot N$

$Var[R] = N g (1-g)$

$$p(R \mid N) = \binom{N}{R} g^R (1-g)^{N-R}$$

(n, r)

$$p(D \mid N) = \sum_{R=0}^{N} p(D \mid N, R)\, p(R \mid N) = \sum_{R=0}^{N} \frac{\binom{R}{r}\binom{N-R}{n-r}}{\binom{N}{n}} \binom{N}{R} g^R (1-g)^{N-R}$$

$$\binom{R}{r}\binom{N-R}{n-r}\binom{N}{R} = \binom{N}{n}\binom{n}{r}\binom{N-n}{R-r}$$

$$= \binom{n}{2} \sum_{R=2}^{N-(n-2)} \binom{N-n}{R-2} g^R (1-g)^{N-R} =$$

$$\sum_{R-2=0}^{(N-n)-(R-2)} \binom{N-n}{R-2} g^{R-2} (1-g)^{(N-n)-(R-2)}$$

$N-R-(n-2)$

$$= \binom{n}{2} \cdot g^2 (1-g)^{n-2} \cdot 1$$

$$\underline{p(R|D,N)} = Const \cdot \binom{N}{R} g^R (1-g)^{N-R} \binom{R}{2}\binom{N-R}{n-2} =$$

$$\binom{N}{n}\binom{n}{2}\binom{N-n}{R-2}$$

$$= Const \cdot \binom{N-n}{R-2} \cdot g^{R-2} (1-g)^{N-R-n+2} \cdot g^2 \cdot (1-g)^{n-2} =$$

$$= Const \cdot \binom{N-n}{R-2} g^{R-2} \cdot (1-g)^{(N-n)-(R-2)}$$

$$E[R|D,N] = 2 + (N-n)g \pm \sqrt{(N-n)g(1-g)}$$

⑤ Энтропия $(p_1, p_2, \dots, p_n)$ „мера неопред."

$$H_n(p_1, \dots, p_n) = -\sum_i p_i \log p_i$$

$$h(mn) = h(m) + h(n)$$

$$h(n) = c \cdot \lg n$$

- H_n - непр.
- $h(n) = H_n\left(\frac{1}{n}, \dots, \frac{1}{n}\right)$ - возрастает по n
- непротиворечивость

$$H_3(p_1, p_2, p_3) = H_2(p_1, p_2+p_3) + (1-p_1) \cdot H_2\left(\frac{p_2}{p_2+p_3}, \frac{p_3}{p_2+p_3}\right)$$

Есть „информация" I, к-рой д-соотв. $(p_1, \dots, p_n)$

$\underline{1}$ $\quad [0 \overset{\delta}{\longrightarrow} 1]$ $\quad \delta = \frac{1}{N}$

N раз бросать честную кость с n гранями

$p_i = \frac{N_i}{N}$ $\quad n^N$ послед. исходов

$$\binom{N}{N_1\, N_2 \dots N_n}$$

$$\Pr[(p_1 \dots p_n)] = n^{-N} \cdot \frac{N!}{N_1!\,N_2! \dots N_n!}$$

$p_1 \dots p_n \to \max$

$$\log N! = N \log N - N + O(\sqrt{N})$$

$$\log \frac{N!}{N_1!\,N_2! \dots N_n!} = N \log N - N - \sum_{i=1}^{n} (N_i \log N_i - N_i) + O(\sqrt{N})$$

$$= \cancel{N \log N} - \cancel{N} - \sum_{i=1}^{n} p_i \cdot N (\cancel{\log N} + \log p_i) + \cancel{\sum_{i=1}^{n} N_i} + O(\sqrt{N})$$

$$= -N \cdot \sum_{i=1}^{n} p_i \log p_i + O(\sqrt{N})$$

⑥ Maximum entropy principle

$$KL(p \| q) = \sum_i p_i \log \frac{p_i}{q_i} \xrightarrow{\min}$$

q_i = const

- кубик с $E[x] = m$

$$H = -\sum_{i=1}^{6} p_i \log p_i \xrightarrow[p_i]{} \max$$

$$p_1 + \dots + p_6 = 1, \quad p_i \geq 0$$

$$p_1 + 2p_2 + \dots + 6p_6 = m$$

$$L = -\sum_{i=1}^{6} p_i \log p_i - \lambda \sum i \cdot p_i - \mu \cdot \sum p_i$$

$$\frac{\partial L}{\partial p_i} = -\log p_i - 1 - \lambda i - \mu = 0$$

$$p_i = e^{-\lambda i - \mu - 1}$$

$$e^{-\mu-1}\sum_{i=1}^{6} e^{-\lambda i} = 1 \qquad e^{-\mu-1}\cdot\sum_{i=1}^{6} i\cdot e^{-\lambda i} = m$$

$$m = \frac{\sum i e^{-\lambda i}}{\sum e^{-i\cdot\lambda}}, \qquad \mu = \log \sum e^{-\lambda\cdot i} - 1$$

(7) <u>Пример MaxEnt</u>: найти $p(x)$, $x\in\mathbb{R}$, доса-макс. энт., с зад. μ, σ^2

$$H[p] = \int_{-\infty}^{\infty} p(x)\log p(x)\,dx \xrightarrow[p]{} \max$$

$$\int p(x)dx = 1, \quad \int x\,p(x)\,dx = \mu, \quad \int (x-\mu)^2 p(x)\,dx = \sigma^2$$

$$L(p,\lambda_0,\lambda_1,\lambda_2) = \int p(x)\log p(x)dx - \lambda_0\left(1-\int p(x)dx\right) - \lambda_1\left(\mu - \int x p(x)dx\right) - \lambda_2\left(\sigma^2 - \int (x-\mu)^2 p(x)dx\right)$$

$p + \delta p(x)$

$$\delta L = \int (\delta p(x))\underbrace{\left(\log p(x) + 1 + \lambda_0 + \lambda_1 x + \lambda_2 (x-\mu)^2\right)}_{=0} dx$$

$0 \quad \forall (\delta p)(x)$

$$0 = \log p(x) + 1 + \lambda_0 + \lambda_1 x + \lambda_2 (x-\mu)^2$$

$$p(x) = e^{-1-\lambda_0-\lambda_1 x - \lambda_2 (x-\mu)^2}$$

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

?$\boxed{\forall q: \quad H(q) \le H(p)}$

$$0 \le KL(q\|p) = \int q(x)\log\frac{q(x)}{p(x)}dx = -H(q) - \underbrace{\int q(x)\log p(x)dx}_{H(q,p)}$$

$$\int q(x)\left(-\frac{1}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}(x-\mu)^2\right)dx =$$

$$= -\frac{1}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\underbrace{\int (x-\mu)^2 q(x)\,dx}_{\mu = \text{"}\mathbb{E}_q[x]\text{"}} = \sigma_q^2 = \sigma^2$$

$$= -\frac{1}{2}\log(2\pi\sigma^2) - \frac{1}{2} = -H[p]$$

$$H[\mathcal{N}(x|\mu,\sigma^2)] = -\mathbb{E}_{\mathcal{N}}\left[-\frac{1}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}(x-\mu)^2\right] =$$

$$= \frac{1}{2}\log(2\pi\sigma^2) + \frac{1}{2\sigma^2}\cdot\mathbb{E}_{\mathcal{N}}\left[(x-\mu)^2\right] = \frac{1}{2}\log(2\pi\sigma^2) + \frac{1}{2}$$

$$0 \le KL(q\|p) = -H(q) + H(p) \rightarrow H(p) \ge H(q)$$