Jeffreys' priors

uninformative priors

① $$p(\theta) = \mathrm{Unif}([0,1]) = \mathrm{Beta}(1,1) \Rightarrow p(\text{open} \mid D) = \frac{n+1}{m+2}$$

$$p(\theta) = \mathrm{Beta}(0,0) = \frac{1}{\theta(1-\theta)}$$

$$\eta = \varphi(\theta) = \log \frac{\theta}{1-\theta}$$

$$\boxed{p(\theta)} = \mathrm{Unif}([0,1])$$

$$\theta = p(\text{open} \mid \eta) = \sigma(\eta) = \frac{1}{1+e^{-\eta}}$$

$$\eta = \log \frac{\theta}{1-\theta}$$

$$\int p_\eta(\eta)\, d\eta = p(A) = \int_A p(\theta)\, d\theta = \int_{\varphi^{-1}(A)} p(\varphi^{-1}(\eta)) \cdot \frac{d\eta}{|\varphi'(\eta)|}$$

$$p_\eta(\eta) = \frac{p_\theta(\theta)}{|\varphi'(\theta)|}$$

$$p_\theta(\theta) = p_\eta(\eta) \cdot |\varphi'(\theta)|$$

$$p_\eta(\eta) = \frac{p_\theta(\varphi^{-1}(\eta))}{|\varphi'(\varphi^{-1}(\eta))|} = \frac{1 \left(\frac{1}{1+e^{-\eta}}\right)}{\frac{1-\theta}{\theta}} =$$

$$= \frac{1}{e^{-\eta}} \cdot \frac{e^{-\eta}}{(1+e^{-\eta})^2} = \frac{1}{(1+e^{-\eta})^2}$$

② Jeffreys' prior $p_\theta(\theta)$ — JP, если для любой монот. ф-ции h, $\eta = h(\theta)$,

$$\boxed{p_\eta(\eta) = \frac{p_\theta(h^{-1}(\eta))}{|h'(h^{-1}(\eta))|}}$$

$$I(\theta) = \mathbb{E}_{x_n \sim p(\theta)}\left[\left(\frac{\partial \log p}{\partial \theta}\right)^2\right]$$

improper priors

$$\boxed{p(\theta) = \text{const} \cdot \sqrt{I(\theta)}}$$

$$\eta = h(\theta)$$

$$I_\theta(\theta) = \int \left(\frac{\partial \log p_\theta(x|\theta)}{\partial \theta} \right)^2 p_\theta(x|\theta)\,dx =$$

$$= \int \left(\frac{\partial}{\partial \theta} \log p_\eta(x | h(\theta)) \right)^2 p_\eta(x | h(\theta))\,dx =$$

$$= \int \left(h'(\theta) \cdot \frac{\partial}{\partial \eta} \log p_\eta(x|\eta) \right)^2 \cdot p_\eta(x | h(\theta))\,dx =$$

$$= (h'(\theta))^2 \cdot \int \left(\frac{\partial}{\partial \eta} \log p_\eta(x|\eta) \right)^2 p_\eta(x|\eta)\,dx =$$

$$= (h'(\theta))^2 \cdot I_\eta(\eta)$$

$$p_\eta(\eta) = p_\eta(h(\theta)) = \text{Const} \cdot \sqrt{I_\eta(h(\theta))} = \text{Const} \cdot \frac{\sqrt{I_\theta(\theta)}}{|h'(\theta)|} =$$

$$= \frac{p_\theta(\theta)}{|h'(h^{-1}(\eta))|}$$

③ Монетка

$$\theta^x (1-\theta)^{1-x}$$

$$I(\theta) = E\left[\left(\frac{\partial \log p_\theta(x|\theta)}{\partial \theta} \right)^2 \right] =$$

$$\frac{\partial (x \log \theta + (1-x)\log(1-\theta))}{\partial \theta} = \frac{x}{\theta} - \frac{(1-x)}{1-\theta}$$

$$= \theta \cdot \left(\frac{1}{\theta}\right)^2 + (1-\theta) \cdot \left(\frac{1}{1-\theta}\right)^2 = \frac{1}{\theta} + \frac{1}{1-\theta} = \frac{1}{\theta(1-\theta)}$$

$$p(\theta) \propto \frac{1}{\sqrt{\theta(1-\theta)}}, \quad p(\theta) = \text{Beta}\left(\theta \,\middle|\, \tfrac{1}{2}, \tfrac{1}{2}\right)$$

④ $\bar\theta, \bar\eta \in \mathbb{R}^d$

$$\boxed{p(\bar\theta) \propto \sqrt{\det I(\bar\theta)}}$$

$$p_\eta(\bar\eta) = p_\theta(\bar\theta) \cdot \left| \det \frac{\partial \theta_i}{\partial \eta_j} \right| \propto \sqrt{\det I(\bar\theta) \cdot (\det J)^2} =$$

$$= \sqrt{\det \Big(\underbrace{J^T \cdot \underline{I}(\bar\theta) \cdot J}_{A_{ij}} \Big)} = \sqrt{\det I(\bar\eta)}$$

$$A_{ij} = E\left[\sum_{k,l} \frac{\partial\theta_k}{\partial\eta_i} \cdot \frac{\partial\theta_l}{\partial\eta_j} \cdot \frac{\partial \log p_\theta}{\partial\theta_k} \cdot \frac{\partial \log p_\theta}{\partial\theta_l} \right] =$$

$$= E\left[\frac{\partial \log p_\eta}{\partial\eta_i} \cdot \frac{\partial \log p_\eta}{\partial\eta_j} \right] = \underline{I}(\bar\eta)$$

(5) Гауссиан $\bar\theta = \begin{pmatrix} \mu \\ \sigma^2 \end{pmatrix}$ $\quad p(\bar\theta) = \text{Const} \cdot \sqrt{\det I(\bar\theta)}$

$D = \{x_1, \dots, x_n\}$

$$I(\theta) = E\left[- \begin{pmatrix} \frac{\partial^2 \log p}{\partial\mu^2} & \frac{\partial^2 \log p}{\partial\mu\partial\sigma^2} \\ -\text{''}- & \frac{\partial^2 \log p}{\partial(\sigma^2)^2} \end{pmatrix} \right]$$

$n \cdot s^2 = \sum x_i^2 - 2\bar x \cdot \sum x_i + \bar x^2 \cdot n$

$s^2 = \frac{1}{n} \sum (x_i - \bar x)^2$

$\bar x = \frac{1}{n} \sum x_i$

$$\log p = \text{Const} - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2} \underbrace{\sum_{i=1}^{n} (x_i - \mu)^2}_{''} =$$

$$n \cdot \mu^2 - 2\mu \cdot \sum x_i + \sum x_i^2 =$$

$$= n\cdot\mu^2 - 2\mu \cdot n \cdot \bar x + n \cdot s^2 + \cancel{2 n \cdot \bar x^2} - \cancel{n \bar x^2} =$$

$$= n \cdot s^2 + n(\bar x - \mu)^2$$

$$= \text{Const} - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\left(n \cdot s^2 + n(\bar x - \mu)^2\right)$$

$$\frac{\partial \log p}{\partial \mu} = + \frac{n(\bar x - \mu)}{\sigma^2}$$

$$\frac{\partial \log p}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{n \cdot s^2 + n(\bar x - \mu)^2}{4\sigma^4}$$

$$\frac{\partial^2 \log p}{\partial\mu^2} = -\frac{n}{\sigma^2}, \quad \frac{\partial^2 \log p}{\partial\mu\partial\sigma^2} = -\frac{n(\bar x - \mu)}{\sigma^4}$$

$''\ \sigma^2 \qquad ''\ \sigma^2 \qquad \frac{n}{2\sigma^4} - \frac{\cancel{2}\cancel{\sigma^2} n}{\cancel{2}\cancel{\sigma^6} 4} = -\frac{n}{2\sigma^4}$

$$\frac{\partial^2 \log p}{\partial (\sigma^2)^2} = \frac{n}{2\sigma^4} - \frac{\overbrace{ns^2}^{E} + \overbrace{n(\bar{x}-\mu)^2}^{E}}{2\sigma^6}$$

$$I(\bar\theta) = \mathop{E}_{x}\left[\begin{pmatrix} n/\sigma^2 & n(\bar{x}-\mu)/\sigma^4 \\ -''- & -n/2\sigma^4 + \text{---}/2\sigma^6 \end{pmatrix}\right]$$

$$I(\bar\theta) = \begin{pmatrix} n/\sigma^2 & 0 \\ 0 & n/2\sigma^4 \end{pmatrix}$$

$$\det I(\bar\theta) = \frac{n^2}{2\sigma^6}, \quad \boxed{p(\bar\theta) = J(\mu,\sigma^2) \propto \frac{1}{\sigma^3}}$$

inverse chi-squared distribution

6) Экспоненциальное семейство

Exponential family: парам.-сем-во $p(\bar{x}|\bar\theta)$ прин. экс.-сем., если

$$p(\bar{x}|\bar\theta) = h(\bar{x}) \cdot e^{\bar\eta(\bar\theta)^T \bar{t}(\bar{x}) - a(\bar\theta)} = h(\bar{x})\, g(\bar\theta)\, e^{\bar\eta(\bar\theta)^T \bar{t}(\bar{x})}$$

для нек-рых $h, \bar\eta, \bar{t}, a, g$ $\qquad g(\bar\theta) = e^{-a(\bar\theta)}$

$\bar{t}(\bar{x})$ – sufficient statistics

$$\mathrm{Binom}(k|n,p) = \binom{n}{k} p^k (1-p)^{n-k} =$$

$$= \binom{n}{k} \cdot e^{k \cdot \log p + (n-k)\log(1-p)} = \binom{n}{k} e^{k \cdot \log\frac{p}{1-p} + n \cdot \log(1-p)}$$

$t(k) = k \quad h(k) = \binom{n}{k}, \quad a(p) = -n\log(1-p)$

$$\boxed{\eta(p) = \log\frac{p}{1-p}} \quad \boxed{p = \frac{1}{1+e^{-\eta}}}$$

log-odds

$$\mathrm{Mult}(\bar{x} \mid n, p_1, \dots, p_k) = \underbrace{\frac{n!}{x_1!\, x_2! \dots x_k!}}_{h(\bar{x})} \cdot e^{\sum_{k=1}^{K} x_k \log p_k} \quad \cancel{-a(\bar{p}) = 0}$$

$$\bar{t}(\bar{x}) = \bar{x} \qquad \cancel{\bar{\eta}(\bar{p}) = \log \bar{p}}$$

$$= h(\bar{x}) \cdot e^{\sum_{k=1}^{K-1} x_k \log p_k + \left(n - \sum_{k=1}^{K-1} x_k\right) \log\left(1 - \sum_{k=1}^{K-1} p_k\right)} =$$

$$= \underline{h(\bar{x})} \cdot e^{\sum_{k=1}^{K-1} x_k \cdot \log \frac{p_k}{1 - \sum \dots} + \underbrace{n \log(1 - \dots)}_{= -a(\bar{p})}}$$

$$t(\bar{x}) = \bar{x}_{1 \dots K-1}$$

$$\underline{\eta_i(\bar{p})} = \log \frac{p_i}{1 - \sum_{k=1}^{K-1} p_k} = \log \frac{p_i}{p_K}$$

$$e^{\eta_i} = p_i / p_K = p_i / \left(1 - \sum p_i\right)$$

$$e^{\eta_i} - e^{\eta_i} \sum_{1}^{K-1} p_k = p_i$$

$$\sum_{1}^{K-1} p_i = \sum e^{\eta_i} - (\ )(\ )$$

$$\sum_{1}^{K-1} p_i = \left(\sum^{K-1} e^{\eta_i}\right) \cdot \left(1 - \sum^{K-1} p_i\right)$$

$$\sum p_i = \frac{\sum^{K-1} e^{\eta_i}}{1 + \sum^{K-1} e^{\eta_i}}$$

$$e^{\eta_i}\left(1 - \frac{\sum^{K-1} e^{\eta_i}}{1 + \sum^{K-1} e^{\eta_i}}\right) = \frac{e^{\eta_i}}{1 + \sum_{j=1}^{K-1} e^{\eta_j}} = \frac{e^{\eta_i}}{\sum_{j=1}^{K} e^{\eta_j}}$$

$$e^{\eta_K} = 1$$

$$\boxed{p_i = \frac{e^{\eta_i}}{\sum_j e^{\eta_j}}}$$

$$\bar{p} = \mathrm{softmax}(\bar{\eta})$$

$$p(x \mid \lambda) = \frac{1}{x!} \lambda^x e^{-\lambda} = \underbrace{\frac{1}{x!}}_{h(x)} e^{\overbrace{x}^{t(x)} \overbrace{\log \lambda}^{\eta(\lambda)} \overbrace{- \lambda}^{a(\lambda)}}$$

$$p(k \mid n, K, N) \propto \binom{K}{k} \binom{N-K}{n-k}$$