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GEORG CANTOR AS THE AUTHOR OF CONSTRUCTIONS PLAYING FUNDAMENTAL ROLES IN CONSTRUCTIVE MATHEMATICS

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An extended version of the author's talk at the meeting of the St. Petersburg Mathematical Society (March 3, 1995), dedicated to the 150th anniversary of G. Cantor's birth, is presented. The following inventions of Cantor and their roles in constructive mathematics are discussed: the system of notation for order-types less than ε_0, a constructive (in essence) definition of the notion of real number, and Cantor's "diagonal" construction. Bibliography: 22 titles.

The 150th anniversary of Georg Cantor, which is celebrated in 1995, is a suitable occasion to recall not only set theory created by him, but also some achievements of the great mathematician, which played roles in the history of science which could not be imagined by Cantor, who was devoted to a set-theoretical "way of mathematical thinking." By this we mean a series of constructions invented by Cantor. Below we discuss only three constructions: the system of notation for order-types of well ordered sets which are less than ε_0 (§1), the constructive (in essence) idea of the definition of a real number (§2), and Cantor's famous "diagonal" construction (§3).

§1

The system of notation for order-types less than ε_0, invented by Cantor, has played an unexpected role in realizing the character of difficulties arising in the analysis of foundations of mathematics ***already on the level of natural numbers***, and turned out to be a source of additional arguments for those mathematicians who undertook, at the end of the nineteenth century and in the twentieth century, a critical analysis of the idealizations occuring in forming the "intuitive base" of set theory. This has been an analysis of the "level of agreement" between the idealizations accepted and the results of experimental study of the nature (on the macro-, micro-, and megalevel of detailing and scope in space-time).

Cantor came to the notions of well ordered set and transfinite ordinal number (ordinal in the modern terminology) by considering the operation assigning every closed set in the number line its derived set (i.e., the set constituted by all its limit points). Cantor has mentally iterated the operation (under the assumption that a certain initial closed set F is given) and added to the iterations the "passages to the limit" by "constructing" the intersection of all sets obtained in the earlier stages. When performing such a "passage to the limit" on the intuitive level, we use an act of imagination, called the abstraction of complete (actual) infinity. The idea of a "natural" relation of preceding for the elements of the mentally constructed family of sets is formed on the same level.

After this "generalized" iteration (where only the first steps have the form of the "usual" induction, and a "far-reaching" generalization of such induction is obtained as a whole), Cantor came to the idea of well ordered sets and their order-types. The corresponding definitions are stated in terms of general set theory, and the rule of transfinite induction is formed in the same terms. In axiomatic set theory, the rule of transfinite induction has the status of a derived rule of inference, and certain statements in various areas of mathematics get the status of theorems only due to this rule of inference.

In constructive mathematics, the rule of transfinite induction cannot even be stated in the "generality" in which it appears in set theory. At the same time, it was Cantor who suggested the construction allowing one to state certain special cases of this rule and discuss them from the point of view of acceptability. In Cantor's theory, the ordinals are thought of (on the intuitive level) as certain "abstract entities," but a notation is introduced for some "entities" of this kind. In particular, one of the order-types is denoted by ε_0, and all order-types preceding ε_0 can also be denoted by bulky strings of symbols of a certain

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form, constructed from the symbols 0, ω, and $+$ according to special generating rules. These strings of symbols can be found in many textbooks. They are *constructively defined objects*, and in the framework of constructive mathematics it is quite sensible to consider them without appealing to ideas of some "abstract entities." For example, to "get" in the intuitive "generating procedure" to the set having the "ordinal index" $\omega^{\omega^{\omega^1+1}} + \omega^1 + 1 + 1$ (here 1 denotes the expression ω^0), it is necessary to "go through" a very complicated infinite hierarchy of infinite procedures. At the same time, "ordinal index" itself can be constructed in ten steps by the corresponding generating rules. After that other authors suggested "well-defined" systems of notation also for "much wider" scales of order-types corresponding to certain well orderings of countable sets. Furthermore, a general notion of *constructible ordinal* has been suggested, but the "explanation" of the definition appeals to ideas of "freely chosen sequences," which are used in intuitionistic mathematics in contrast to mathematical theories belonging to the constructive direction in mathematics.

Cantor's system of notation for the ordinals less than ε_0, as well as the above-mentioned "well-defined" extensions of this system of notation admit "rectilinear" versions (using certain additional elementary symbols), and it is not difficult to pass from them (using appropriate algorithms which one-to-one map the set of all strings of symbols of the given notation onto the set of all natural numbers, so that zero is mapped to zero) to *using the natural numbers as the constructive ordinals of the scale.* Here the term *natural number* is used in the finitary sense. We mean the version where this term is used as common name of the words 0, 0 |, 0 | |, 0 | | |, ... The ordering which is introduced for the objects of the notation considered can be "carried over" onto the natural numbers, and if the type of ordering is denoted by a certain symbol α (which certainly does not belong to the notation under consideration), then the corresponding relation of preceding is denoted by $<_\alpha$. For the "well-defined" scales of constructible ordinals (in particular, for Cantor's scale) the relation of preceding is expessible in primitive recursive arithmetic and hence is algorithmically verifiable.

Hilbert [1, 2] drew the attention of mathematicians to the merits of constructing elementary arithmetic and algebra in such a way that the terms *natural number*, *integral number*, *rational number*, *polynomial with integral coefficients*, etc. denote *strings of symbols* of special types characterized by generating rules (and not some "abstract entities"). The further examples of *constructively defined objects* (CDO), which drew Hilbert's attention, are formulas of logical languages, inferences in logical calculi, etc. Kronecker established the possibility of definition of the notion *algebraic number* which does not use the notion of real number and characterizes algebraic numbers as CDO (van der Corput [4] suggested a "more visual" definition).

After singling out CDO as specific objects of study, Hilbert "outlined" (a sketch of) a specific way of considering them, which he called *finitary attitude (finitary point of view)* [1, 2, 3]. The origins of this attitude can be found in certain statements of Kronecker. The finitary attitude suggests considering CDO as *visually conceivable* ("almost physical") objects by using the idealization which is called the abstraction of potential infinity (potential realizability) and not the abstraction of complete (actual) infinity. In so doing "... one uses direct contensive arguments, which are performed as mental experiments over visually conceivable objects and do not depend on assumptions of axiomatic character." (See [2].) Under such an attitude, the problem of explaining mathematical assertions about CDO turns out to be the *problem of extrapolation* to such "almost physical" objects of those ideas of true judgments that are formed when the "natural structures," which are discretely structured and can be viewed in finite time, are experimentally studied on the macroscopic level of their detail and "scope" in space-time; such structures are amenable to satisfactory simulation in mathematical theories with finite sets of objects of study.

In mathematical theories of this kind, all statements which are formulated in the languages "usual" for such theories admit clear semantics of operational character: the logical signs $\neg$, &, $\vee$, $\rightarrow$, $\leftrightarrow$ are understood as designations of the corresponding Boolean functions, and formulas of the form $\forall x A$ and $\exists x A$, where A is a quantifier-free formula and admissible values of the variable x are constants $c_1, \dots, c_N$, are interpreted as $(A_1 \& \dots \& A_N)$ and $(A_1 \vee \dots \vee A_N)$ respectively. Here A_i denotes the result of substituting the constant c_i for all occurences of the variable x in A $(1 \le i \le N)$.

When attempting to extrapolate this semantics to arithmetic (to the theory of natural numbers), crucial difficulties arise, and a very important role in realizing their character was played by the scale of constructible ordinals, invented by Cantor. First consider statements of the two following kinds S_1 and S_2 (below, the term *recursive function* is used as a synonym of the term *partially recursive function*):

S_1. *The recursive arithmetical function f of n variables is total.*

Symbolically this can be written in the form

$$\forall x_1 \ldots \forall x_n ! f(x_1, \ldots, x_n).$$

Here $x_1, \ldots, x_n$ are the variables for natural numbers and by writing $!f(x_1, \ldots, x_n)$ we mean *the process of computing the value of $f(x_1, \ldots, x_n)$ terminates.*

S_2. *The total recursive function ϕ of n variables is a 0-function.*

Symbolically this is written as follows:

$$\forall x_1 \ldots \forall x_n (\phi(x_1, \ldots, x_n) = 0).$$

It is impossible to consider these statements as claims about some phenomena in the "world of experimental data," because the ideas of "very large" natural numbers are formed on the basis of "experimenting" with strings of symbols which already exist or are immediately created, and using the abstraction of potential realizability, analogies, etc. In addition, these ideas actually appeal to hypotheses of global cosmological character. For statements of type S_1 and S_2 the finitary attitude suggests looking for those *extrapolations* of the notion of *true statement* used in mathematical theories with finite sets of objects, which are characterized by "sufficient conditions of truth" (considered "against the background" of the abstraction of potential realizability) such that their fulfillment, when an n-tuple $M_1, \ldots, M_n$ of natural numbers is constructed, gives a "sufficient reason" for recognizing as true that special case of the considered "generalizing" statement of type S_1 or S_2 which corresponds to the n-tuple constructed, true in the sense which is natural (and obvious) for the obtained "particular" statement *containing no variables.* The difficult component of the process is motivation of "acceptability" of the extrapolation suggested. When such extrapolation is sufficiently clearly characterized, it can be given the status of *partial semantics* open for "weakenings," and in this situation the notion of *true statement* is naturally identified with that of *statement having justification under given particular semantics.* The "range of types" of the semantics which are actually used in mathematics is very wide, and the above-mentioned Cantor scale of ordinals made a fundamental contribution to realizing the complexity of the resulting situation.

If f is a concrete *primitive recursive* function, then the particular semantics under which the justification of statement S_1 can be "obtained" is based on the "usual" arithmetical induction (corresponding to the ordering of the natural numbers modeled on the ordinal ω).

Now let us consider a more complicated situation. Let N be a natural number, g a primitive recursive function of two variables, α a certain type of well-ordering of natural numbers such that the relation of preceding $<_\alpha$ is "well defined" (e.g., $\alpha = \varepsilon_0$), and γ a primitive recursive function such that $\gamma(0) = 0$ and $\gamma(x \mid) <_\alpha x \mid$ for every natural x. Let f be the recursive function characterized by the following conditional equalities (giving an example of defining a function by ordinal recursion under the ordering of natural numbers by the ordering $<_\alpha$):

$$f(0) \simeq N, \qquad f(x \mid) \simeq g(x \mid, f(\gamma(x \mid))).$$

Under the standard procedure of computing the value of f on the given positive number M_0, the expressions $f(\gamma(M_0))$, $f(\gamma(M_1))$, $f(\gamma(M_2))$, $\ldots$, where $M_{i+1} = \gamma(M_i)$, occur "inside" the consecutively obtained expressions, and hence $M_{i+1} <_\alpha M_i$ for $i \geq 0$. The process terminates if at some step the expression $f(0)$ is obtained, and it does not if "in the process of transfinite descent" from the number M_0 via the function γ zero is not obtained. Thus, a substantive justification of totality of f includes a substantive justification of the following claim:

S_3. *Whatever the natural number x is, in the process of "transfinite descent from it" via the function γ zero is obtained.*

A similar situation arises when considering statements of the form S_2. Let us assume (for example) that ϕ is a primitive recursive function of one variable and the symbol γ has the previous meaning. The rule of inference

$$\frac{\phi(0) = 0, \qquad \phi(\gamma(x \mid)) = 0 \to \phi(x \mid) = 0}{\phi(x) = 0} \tag{$*$}$$

is the rule of transfinite induction under the ordering of natural numbers modeled on α with descent function γ. The justification of its "acceptability" includes the statement S_3.

Gödel [5] says the following about the justification of the latter statement: "... surely, the justification (die Gültigkeit) of inference by reduction to ε_0 cannot be made intuitively obvious, which is possible, say, for ω^2. More precisely, this means that we already cannot consider all different structural possibilities, which exist for decreasing sequences, and thus have no possibility to verify visually that every such sequence necessarily terminates. ... by gradually passing from smaller to larger ordinal numbers, such *visual knowledge* cannot be realized, and only abstract perception using notions of higher levels is possible."

The idea of "obviousness" of statement S_3 in the case where $\alpha = \varepsilon_0$ (and in general for any constructible ordinal α) can arise due to the fact that in the axiomatical set theory one can prove that the relation $<_\alpha$ is a well-ordering in the sense of set theory. Nevertheless, in the analysis of foundations of arithmetic the possibility of finitary (axiom-less, "contensive") justification of S_3 is discussed, and in such formulation of the question the situation turns out to be very complicated.

The statement on the consistency of Peano's arithmetic (and any other mathematical theory having the form of a calculus in the modern sense of the word) can be expressed by a formula looking like $\forall x(\phi_0(x) = 0)$, where ϕ_0 is a certain primitive recursive function (depending on the specific calculus). The first justification of consistency of Peano's arithmetic was suggested by G. Gentzen [6, 7]. First of all, it contains an indication to the particular semantics for statements of type S_2 under which the considered statement $\forall x(\phi_0(x) = 0)$ admits a justification. In this particular semantics, inference $(*)$ for $\alpha = \varepsilon_0$ is considered as an acceptable tool of extrapolation of ideas of true statements. To justify the acceptability, Gentzen proposes an argument using ideas of "attainability" of ordinals, but these ideas go far outside the limits of ideas which can be explained by appealing to the "usual" arithmetical induction. Gentzen notes in [7]: "However, I cannot indicate the "place" where ... the induction which is undoubtedly acceptable from the constructive point of view ends and *doubtful* transfinite induction begins."

Already before Gentzen's above-mentioned paper, a statement of the type under consideration was constructed which admits no justification in the framework of semantics with "usual" induction, but at the same time there is no necessity to appeal to transfinite induction. Here we mean Gödel's famous example of a statement such that the translations of statements G and $\neg G$ into arithmetical language are not deducible in this axiomatic theory, but G itself has a contensive justification of special type, based on understanding the "specific meaning" of the statement.

Thus, Gödel and Gentzen actually ascertain the impossibility of any "full and final" refinement of ideas, suggested by concrete examples, concerning possible extrapolations of the notion of true statement, used in theories with finite sets of objects, to statements S_1 and S_2. In this sense, the problem of formulating a "precise" semantics for such statements can be characterized as a dead end. This stimulates choosing particular semantics for mentioned statements (and for the statements which can be reduced to them by using clear semantic agreements) and "working" with statements which are true in the semantics chosen. In the framework of this approach one finds useful (as a suitable base for developing various theories of finitary mathematics) primitive recursive functions, the *quantifier-free* language based on them (it uses the relation $=$ and Boolean functions), and the apparatus of logical inference compatible with "usual" arithmetical induction. This entire complex acquired the name *primitive recursive arithmetic* (PRA). When using a formula of PRA as a statement, it is implied that in front of the whole formula there are "invisible" universal quantifiers connecting all variables occuring in the formula.

All mentioned above stimulates the discussion of one more question concerning the "intuitive base" of Cantor's set theory. We restrict ourselves by considering the set af all natural numbers and its subsets which admit a definition of the form

$$m \in M \leftrightarrow \forall x(\psi(m, x) = 0), \qquad (**)$$

where ψ is a primitive recursive function of two variables. The condition written to the right of $\leftrightarrow$ is "expressible" in the language of set theory. Subsequently substituting the numbers 0, 0 |, 0 | |, 0 | | |, ... for the variable m in this condition, we will obtain statements for which the question of truthfulness becomes sensible only after fixing a specific particular semantics (the exceptions are the cases where the mentioned condition can be checked algorithmically). When changing the values of the variable m one may want to vary the particular semantics. Thus, even for the sets characterized by definitions of type (**), the "contensive explanation" of the relation $\in$ turns out to be "vague." At the same time, "the set of all subsets of the set of positive integers" is considered in the "intuitive base" of set theory as a "well-defined" object. The situation gives an additional reason for a "cautious" attitude to this base.

§2

The definition of the notion of *real number* suggested by Cantor and (independently) by Ch. Merè is based on a certain construction. Here we use the term "construction" not in the sense in which it is used in constructive mathematics, but in a wider sense. The discussed definition appeals to "almost physical" ideas of sequences of rational numbers as of processes of consequently constructing some objects, and from this point of view it is "intuitively constructive." "Algorithmization" of this construction supplemented with appropriete modifications and generalizations, opened the way to develop constructive and, what is more, *finitary* (in the sense of Hilbert) versions of mathematical analysis.

Finitary versions of mathematical theories (and in some respects also versions that are constructive in the wide sense) yield a "positive" continuation of critical analysis of Cantor's set theory: they are suggested as alternatives to the versions using set theory. Such alternatives give certain answers to the question: whether construction of mathematical theories without using the set-theoretical "way of thinking" is possible, and what are the merits of such a construction. A historical paradox is that Cantor was the author of the idea which (combined with other ideas) turned out to be a suitable tool for creating altenatives to the "mathematical world-outlook" which is his child.

Defining real numbers as fundamental sequences of rationals (considered "against the background" of a certain equality relation), suggested by Cantor and Merè, competed with the definitions proposed by Weierstrass and Dedekind. However, it was their definition which was very fruitfully generalized to the notion of a fundamental sequence of points of a given metric space, proposed by Hausdorff. The version of this definition convenient for "constructivization" has the following form:

Let $\langle \mathcal{M}, \rho \rangle$ be a metric space (with support $\mathcal{M}$ and metric function ρ) and let F be a map of type $\mathbb{N} \mapsto \mathcal{M}$. We say that F is a *fundamental sequence of points* of this space if there exists a function h of type $\mathbb{N} \mapsto \mathbb{N}$ which is *a regulator of convergence in itself* for the sequence F of points, i.e., for every three natural numbers i, k, l the condition

$$(k \geq h(i) \;\; \& \;\; l \geq h(i)) \rightarrow \rho(F(k), F(l)) < 2^{-i} \qquad (***)$$

is fulfilled.

If a fundamental sequence F is "given" but its convergence regulator is not provided, then there arises a "creative" problem of finding, say, a natural number N such that for every k the distance from $F(0)$ to $F(k)$ is less than N. Such insufficiency of information contained in the fundamental sequence F itself (which is especially "keenly" felt in the system of notions of constructive mathematics) stimulates preferring the completions of metric spaces, consisting of *duplexes* (i.e., the pairs of the form $\langle F, h \rangle$ satisfying condition $(***)$ for any i, k, l) to those consisting of fundamental sequences of points.

An analog of such completion in the system of notion of constructive mathematical analysis results from restricting the "field of vision" by constructible metric spaces, algorithmically defined sequences of points and algorithmically defined convergence regulators (see, for example, [9; §10]). However, the criticism of the "intuitive base" of set theory, discussed in §1, can be extended to the "intuitive base" of traditional mathematical analysis: the statements formulated in the latter go far from the limits of the language of finitary mathematics and hence suggest guessing difficult "semantic riddles." In particular, one of the central notions, that of constructible function on constructible continuum (see [10]), is characterized by a condition of the form $\forall x \neg \forall y \neg \forall z (\lambda(c, x, y, z) = 0)$, where λ is a primitive recursive function, c is the arithmetical code of the constructible function under consideration, and x, y, z are variables for natural numbers. This is the arithmetical version of the condition, resulting from suitable coding by naturals of constructible objects of different types, occuring in the definition of the notion.

It is said in monograph [2] that in mathematical analysis "... nonfinitary ways of creating notions and nonfinitary proofs are an integral part of the theory." However, when discussing the porоblem of "finitarization" of mathematical analysis we mean not one or another version of "retelling" definitions and assertions of the standard mathematical analysis in the language acceptable in finitary mathematics (such attempts are hopeless!), but using the means of that language for construction of a system of notions which *may differ very much from the traditional one* but gives a possibility to develop the *technical part* of mathematical analysis to the extent which is "practically sufficient" for its traditional applications and (which is desirable) in a form giving some additional possibilities (for example, a possibility to clearly "see" the types of constructions occuring when the solution of the considered problem is constructed, and the types of initial data for such constructions). From this point of view, we can certify that Hilbert's opinion that mathematical analysis cannot be "finitarized" has been disproved in the course of further development of mathematics.

The process of "finitarization" was started in works of R. L. Goodstein [11, 12], where he introduces and studies *approximatively defined functions* of some types, which are finitary analogs of continuous (or continiously differentiable) functions on a closed rational segment. These analogs are remarkable in that they require no preliminary introduction of the notion of a real number in any form. The functions introduced by Goodstein are (in essence) objects of a special type corresponding to the following definition by Euler: "A function of varying quantity is an analytic expression composed in some way from the varying quantity and numbers or constant quantities." The system of notions suggested by Goodstein turned out not to be sufficiently suitable for extending it to various functional spaces, and for this reason some modifications and extrapolations of the system were proposed.

The choice of basic notions for satisfactory construction of finitary versions of several areas in mathematical analysis was prepared (first of all) by specific theorems from different divisions of mathematical analysis, having the following form: "In the metric space $\langle\mathcal{M},\rho\rangle$ there exists a countable everywhere dense set." Theorems of this kind are usually considered as establishment of a very essential property of the considered space, providing a possibility (if $\langle\mathcal{M},\rho\rangle$ is a complete space) to introduce into consideration the "isometric double" of the given space, obtained as the result of completion of a certain countable metric space. However, in traditional mathematical analysis "it is not customary" to follow the example of Cantor and Merè and use "the completion procedure" as a tool systematically applied for defining mathematical objects of "complex" type (in particular, "complex" elements of functional spaces) based on objects of simple types. But precisely following this example opens the way to overcoming the "taboo" put by Hilbert onto creation of finitary versions of different areas of mathematical analysis.

In many cases, traditional proofs of specific theorems on existence of countable everywhere dense subsets in metric spaces are (or can easily be made) proofs of some detailed versions of the considered theorems, which establish a series of essential features of the countable everywhere dense subsets presented in the proofs and have the following form:

> "In the metric space $\langle\mathcal{M},\rho\rangle$ there exists an everywhere dense subset T such that (1) its elements are individually given as constructively defined objects of a certain specific type (we call them objects of type τ), and each object of type τ "defines" a certain element of the set T, (2) the objects of type τ constitute a decidable and hence enumerable set (we can assume essentially without loss of generality that the objects of type τ are words in a suitable alphabet and that we are dealing with decidability of the set of objects of type τ as the subset of the set of all possible words in the alphabet), (3) the metric function ρ on the set T considered as the set of objects of type τ is given as an algorithm, and (4) if X and Y are objects of type τ, then $\rho(X,Y)$ is a rational (in the second version, algebraic) number."

In specific functional spaces, the elements of the chosen countable everywhere dense subset are (for example) polynomials with rational coefficients, polygonal or step-functions with rational coordinates of "vertices." In the metric space constituted by the subsets of the real line which are Lebesgue measurable and have finite measure they are finite collections of intervals with rational ends, and in the space of completely continious operators defined on the space of functions of square integrable functions on the interval $[0,1]$ they are finite-dimensional operators given as square matrices (of arbitrary size) with rational (algebraic) elements.

Any particular theorem of the above-described kind "hints" at a specific point of view at the metric space under consideration: the elements of $\langle\mathcal{M},\rho\rangle$ are "almost CDO of type τ." This point of view is a guiding line when defining finitary analogs of many specific metric spaces playing paramount roles are in mathematical analysis and its applications (including the analogs of specific functional spaces, spaces of operators of certain types "acting" in specific functional spaces, specific metric spaces consisting of objects of geometric character, etc.). Here we mean the *definitions based on the idea of completion* of a suitable "elementary" metric space consisting of CDO of a certain type τ and satisfying conditions (1)–(4). Certainly, the objects added to the initial "elementary" space acquire definitions in the framework of finitary mathematics, and a finitary analog of one or another metric space $\langle \mathcal{M}, \rho \rangle$ from the arsenal of traditional mathematical analyses, which is created in this way, can "exist independently" without appealing even to the idea of the definition used for *initial* characterization of elements of $\mathcal{M}$ (for example, for the space of Lebesgue integrable functions on the interval $[0, 1]$ this is the idea of "a Lebesgue measurable function defined on all real numbers from the interval $[0, 1]$."

The constructive direction of mathematics contains mathematical theories of finitary type (characterized by very "severe" requirements to "clarification" of formulated statements and definitions), as well as the theories belonging to "wide" constructivism. The latter contain not only the statements which are explained on the level of requirements of the finitary attitude (which perhaps is more widely understood than originally conceived), but also statements on CDO with complex combinations of quantifiers and propositional logical connectives, causing semantic problems (here in general, statements of types S_1 and S_2 are understood as "immediately clear"). Theories of finitary type are also theories of the second type since the wide "constructivism" has no grudges of "disproving" character to the theories of finitary direction.

In the version of mathematical analysis developed in the framework of "wide constructivism" (see [10, 13]), the objects added to the "elementary" metric space under its constructive completion are duplexes of the form $\langle F, h \rangle$, where F and h are algorithms which are total in a "immediately clear" sense and satisfy "immediately clear" quantifier-free condition $(***)$. In some cases only the duplexes satisfying an appropriate special condition are added (see [15]).

The passage from the described version of completion of the "elementary" space to *finitary completion* requires preliminary specification of some *particular finitary mathematics* taken as the base for construction of specific mathematical theories. This step was outlined in §2, some details can be found in [16]. For technical reasons it is appropriate to code by natural numbers the CDO of different types occuring in the considered theory, and to use the arithmetical recursive functions as algorithms of standard type. By this approach the above-mentioned PRA turns out to be a convenient (and "practically sufficient" in a wide range of cases) particular finitary arithmetic (see [2], [17], or [18]).

By finitary completion of "elementary" space, the objects added to it are duplexes of the form $\langle F^*, h \rangle$ such that (a) F^* and h are arithmetical recursive functions from that class of recursive functions whose totality can be justified under the semantics of chosen particular finitary arithmetic, and (b) using the technique of inferences of the chosen particular finitary arithmetic one can prove the statement expressed by the quantifier-free arithmetical formula resulting from formula $(***)$ under its natural (w.r.t. the coding of CDO) "translation" into the quantifier arithmetical language (in the process of "translation," F is replaced with F^*, which corresponds to replacing a sequence of points of the "elementary" space with the sequence of the arithmetical codes).

There are cases when it is necessary to change the mentioned version of finitary completion of "elementary" spaces in some detail (for example to satisfy condition (4), sometimes it is appropriate to convert the metric space into a countably metric space equivalent to it in the sense of uniform topology, etc.). To obtain "finitary expressions" for some notions and theories of the traditional version of mathematical analysis which are worthy of notice, "quasifundamental" sequences of points of metric spaces are introduced (this idea goes back to a property of monotone and bounded sequences of rational numbers, which was noted by E. A. Bishop [14; p. 109]; the idea was worked out in [19]).

All of the preceding concerned finitary analogs of separable metric spaces. The space of generalized functions in the sense of Sobolev–Schwartz yields an example of nonseparable topological space for which, already in the framework of set-theoretical mathematics, the basis idea of Cantor and Merè directed the way to creating a version which is much more "visible" than the original one. In [20, 21] this idea has been subjected to certain changes which allowed one to adjust it to the radically new situation and to obtain an approximative version of notion of generalized function. Now it is not difficult to pass from this version to a finitary analog of notion of generalized function (see [22]), where the original "structural material" is polynomials with rational coefficients.

Finitary versions of different divisions of mathematical analysis (and other areas of mathematics) provide mathematical applications with means for constructing theoretical models of various fragments of the "world of experimental data," which have "purely informational character." By this we mean the models in which the implied objects and their interrelations are *individually presented* in sufficient detail and with satisfactory accurateness by "concrete pieces of information" having the form of discrete symbolic constructions. These "concrete pieces of information" are considered with constant regard for this feature of theirs and with using no idealizations without which we are able to do. We also do not "surround" them by any "ideal objects" having no individual definitions via symbolic constructions. Such purpose forms the *finitary type* of mathematical thinking, whose highlights were outlined by Kronecker and Hilbert. In certain respects, this gives an alternative to the *set-theoretical type* of mathematical thinking, introduced into the mathematical use by Cantor. However, the power of mathematical intuition characteristic for Cantor has manifested in that his summary contribution to science was of much help for beginning and development of a promising alternative to the "mathematical world-outlook" chosen by him.

§3

Cantor invented the *diagonal construction*, a method of "obtaining" from any sequence of reals a real which is different from all members of the sequence. This method becomes an algorithm in an obvious way if the members of the sequence are real duplexes and the sequence represents an algorithm. Various modifications, analogs, and generalizations of the diagonal construction in the framework of set-theoretical mathematics have proven to be very productive tools, especially in those cases when a classification of certain objects is considered and it is required to "obtain" an object of one or another type, which does not occur in the classification. In constructive mathematics, the role of the *diagonal construction* is still more significant, since it interacts with other algorithmic constructions and provides the means for discovering numerous "deeply hidden" properties of algorithms and enumerable sets. In particular, one of the most profound theorems of the theory of algorithms, the theorem on the fixed point of a recursive operator, which was stated and proved by S. Kleene, "comes back" to the idea of Cantor's *diagonal construction*.

Constructions of "diagonal type" are widely used in the finitary version of mathematical analysis. Let me mention only one example.

In mathematical applications, experimentally interpreted mathematical objects often turn out to be *interval functions* "originating" from functions of real variable and corresponding to these or those ideas of the "approximately defined" values of the considered functions at the "approximately defined" values of variable. In relation to interval functions, the initial functions play the role of "tools" for the *indirect representation*. In the finitary version of mathematical analysis, interval functions are first defined for the elements of the "elementary" functional space whose finitary completion yields the space of interest for us (§2). For example, if the polynomials with rational coefficient are considered as elements of the space of continious (or continiously differentiable) functions on the interval $[0,1]$, then the "suitable" interval function is (for example) the function under which the triple $\langle P,\sigma,n\rangle$, where P is a polynomial with rational coefficients, σ is a rational interval contained and relatively open in $[0,1]$, and n is a natural number, is assigned the rational interval (A,B) where A (resp., B) is the approximation (obtained by a fixed method) of the algebraic numbers $\min_\sigma P$ (resp., $\max_\sigma P$) from below (resp., from above) by rational numbers with an accuracy of 2^{-n}. (Introduction of n as an additional input data is convenient from the technical point of view.) On the other hand, if the mentioned polynomials are considered as elements of the space of square integrable functions on $[0,1]$, then the "suitable" interval function will be the function under which the same input data are assigned the interval $(C-2^{-n},C+2^{-n})$ where C is the mean integral value of the polynomial P on σ. If W is the duplex added to the "elementary" space under its finitary completion in the first (second) case, then extrapolation of the first (second) interval function onto the triple $\langle W,\sigma,n\rangle$ is performed by means of an appropriate construction of "diagonal type."

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