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On a finitary version of mathematical analysis ☆

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Abstract

An approach to constructing counterparts of some fields of mathematical analysis in the frames of Pilbert's "finitary standpoint" is sketched in this paper. This approach is based on certain results of functional spaces theory development in classical mathematics. © 2002 Elsevier Science B.V. All rights reserved.

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Contents

1. On basic ideas of finitary mathematics

1.1. A prevailing opinion among mathematicians at present is that Cantor's set theory (CST) in its modern form (assuming in particular modern construction of mathematical

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logic) is able to play a role of an intuitively acceptable base of various areas of mathematics, among them that of the mathematical analysis (MA).

On the other hand, basic ideas of the set theory long ago became the objects of a critical analysis. Essentially, this was an analysis of idealizations participating in forming an intuitive foundation of set theory (in the first place the idea of "infinite totalities of simultaneously existing objects") from the viewpoint of the 'level of agreement' of accepted idealizations with the results of experimental investigation of nature on macro-, micro- and mega-levels of detalization and of 'extension' in space and time.

The origins of a critical attitude towards the intuitive basis of CST can be found in the mode of thought of some ancient thinkers, who did not accept the idea of a "finished infinity". (Sometimes, in Russian language literature, one uses a term "actual infinity", but it is less apt.) The development of such views was helped by the absence among the results of an 'experimental investigation' of nature (at least at the level of detalization and extension in time-space available at that time) any data witnessing the existence in nature of at least one 'realization' of the idea of *infinite* totalities of simultaneously existing objects of any distinctly characterized type. A contribution was made also by the appearance of paradoxes 'at the level of pure thought'. In particular, it was difficult to assimilate the idea that a line segment of a positive length 'consists' of separate points. At the same time 'intuitive motivations' of some arguments used, for example, in proofs of geometric theorems appeal in fact to 'forbidden' ideas (in particular, via application of the law of excluded middle).

Refuting the idea of "finished infinity" ancient mathematicians and philosophers recognized as admissible the ideas of a "potential infinity". The latter arises from considering some processes 'developing in time' as well as some procedures. For example, in the first book of Euclid's classical "Elements" the second postulate is stated as follows: "…a restricted straight line can be extended along the straight lines".

1.2. An even more complicated situation arose at the time of formation of the differential and integral calculus. In [7, Introduction] it was characterized as follows:

"Logically precise reasoning, starting from clear definitions and noncontradictory, 'evident' axioms, seemed immaterial to the new pioneers of mathematical science. In a veritable orgy of intuitive guesswork, of cogent reasoning interwoven with nonsensical mysticism, with *a blind confidence in the superhuman power of formal procedure* (the stress is mine, N.Sh.), they conquered a mathematical world of immense riches. Gradually the ecstasy of progress gave way to a spirit of critical self-control. In the nineteenth century the immanent need for consolidation and the desire for more security in the extension of higher learning $\langle\ldots\rangle$ led back to a revision of the foundations of the new mathematics".

The revision of foundations started by Bolzano, Cauchy and Weierstrass prompted a search for basic ideas (in the first place in mathematical analysis) which were able to satisfy to some degree a need for 'clarity'. The system of ideas suggested by Cantor and using the idea of "infinite totalities of simultaneously existing objects" as an intuitive base attracted the attention of many mathematicians as one of the versions claiming to satisfy such a need. The need for 'clarity' was so urgent that many mathematicians agreed to disregard that this version actually rejects the results of the deep analysis of the problem performed by ancient thinkers, and suggests the formation of basic ideas starting from very idealized and superficial analogies with ideas which are formed when one considers 'visualizable' finite totalities of real objects and operators with such totalities. (Appendix A.)

1.3. Already at the very beginning of dissemination of CST to new areas of mathematics the question of 'a level of fantasy' of some abstractions used in CST, the question of the 'level of imagination arbitrariness' acceptable when they are formed, again became a subject of discussion. (Appendix B.)

This discussion which began at the end of the XIX century and the beginning of the XX century essentially differed from the discussions of the finished infinity in the previous ages: 'main characters' of the new discussions tried to complement their criticism by *search and proposals of alternatives* (in the first place the alternative foundations of mathematical analysis). Participants of such a two-sided activity were (in the first place) outstanding mathematicians of the end of the XIX century and the first half of the XX century, L. Kronecker, A. Poincare, L.E.J. Brower, H. Weyl, D. Hilbert (at the middle of the XX century this list was continued by A.A. Markov) who fruitfully worked also on problems of theoretical mechanics and physics and inevitably approached an analysis of foundations of mathematics with 'impressions' of the participants of the process of cognition of nature.

The general direction of the search for alternatives was suggested by the fact that typical problems of applications of mathematics have both as data and potential solutions some 'concrete informations' about some objects (in the wide sense) or some connections between them given as combinations of signs of definite types (for example as words in a suitable alphabet or as discrete 'sign constructions' admitting 'individual specification' by words in a suitable alphabet). In the case of varying initial data of this character one usually looks for an algorithm transforming initial data into required solutions.

Intuitively perceived *optional character* of 'far-reaching' idealizations of CST, in the problems mentioned above directed the thought to a search of the versions of various mathematical theories where the objects are individually specified by *sign constructions* of a suitable type. The latter are 'almost physical' objects, and in their theoretical consideration it is expedient to avoid 'far-reaching' idealizations.

Both purely theoretical considerations arising in critical analysis of intuitive foundations of mathematics and the above-mentioned stimuli arising immediately from applications of mathematics created preconditions for the rise and development of the *constructive direction in mathematics* (*constructive mathematics*). (Appendix C.)

A general feature of specific theories of constructive mathematics is the fact that all objects (of all types) considered in these theories are constructively defined objects. There are, however, considerable (and even principal) differences between various theories in the 'level of demands' to *semantic clarity* (to *definiteness of meaning*) of statements and definitions.

1.4. Even mathematical theories having finite individuum domains (such theories are included in constructive mathematics in an obvious way) rely on mental considerations of 'very extensive' finite totalities on extrapolations and idealizations whose relations with results of physics and cosmology are far from simple. An activity in a 'world of experimental data' on a macroscopic level of detalization and 'embrace', in the space–time of fragments of this 'world', together with the perception of various mentally selected 'actually visualizable' *finite* totalities consisting of *simultaneously existing* objects 'practically unchanged in time', as well as various operations on such totalities leaving elements of these totalities practically unchanged, form the basis of idealized extrapolations 'by an analogy' to ideas of arbitrary (including 'very extensive') finite totalities.

These extrapolations in fact use a number of ideas which are drastically different from modern ideas of physics and cosmology (in particular the idea of *simultaneity* of two spatially remote events which are elements of a finite set under consideration is borrowed from Newtonian mechanics).

Situations in which a cognizing subject arrives as a result of mental separation (maybe using idealized extrapolations mentioned above) of a finite family of finite individuum domains and a finite family of operationally characterized procedures (the term "procedure" is used here and below as a common name for objects operations and predicates) which are totally defined for all data admissible under given types of the procedures, are referred to as *completely finitary situations*. Theoretical models obtained by significant modeling of the completely finitary situations are referred to as *completely finitary mathematical theories*.

Such theories usually use traditional first order logical language, propositional logical signs, $\neg, \&, \vee, \rightarrow, \leftrightarrow$ are understood as designations of suitable Boolean operations, and every formula containing quantifiers (signs $\forall$ and $\exists$) is 'deciphered' as a quantifier-free formula using the following agreements: every formula of the form $\forall xA$ (of the $\exists xA$) where x is an individuum variable and A is a quantifier free formula, is 'deciphered' as a formula $(A_1 \& A_2 \& \cdots \& A_k)$ [respectively, as a formula $(A_1 \vee A_2 \vee \cdots \vee A_k)$], where A_i $(1 \leqslant i \leqslant k)$ denotes a result of substituting in A for the variable x the individuum constant having the number i in the list of all admissible values of the variable x.

In totally finitary theories every closed formula (every sentence) can be perceived as a coded description of a definite (generally multi-step) process or a certain experiment, which is called in the logical literature the process of determining *the logical value of a sentence*. Recursive definitions of the logical value of sentences frequently stated in logical literature 'hint' at methods of development of the deciphering processes, but at the same time they 'shade' a possible immediate description. An immediate and clear description of the procedure takes the form of successive advancement through the syntactic analysis tree of a given sentence from the 'leaves' of the tree to its 'root' performing the basic procedures in order. (It is understood that the sentence considered is preliminarily ‘deciphered’ into a quantifier-free formula.) When the result of such a process is a standard ‘recognition symbol’ in the form of the propositional constant **t** (or just the word “yes” in everyday version), the sentence considered is called *a true sentence*. Hence for completely finitary theories with ‘not too extensive’ individual domains (such that there are reasons to stay at the macroscopic level of consideration of the whole ‘picture’) it is possible to achieve relative clarity in making precise ‘everyday’ intuitive ideas of true sentences [46].

1.5. Transcending the framework of completely finitary mathematical theories but remaining in the framework of theories which study only constructively definable objects and admit some additional, but ‘relatively careful’ extrapolations and idealizations, we pass to the ‘level’ of the theories of *finitary mathematics*. ‘Relative care’ means at least rejection of ideas of “finished infinity” and admission of only logico-mathematical languages where sentences admit explanation by ‘relatively visualizable’ extrapolation of semantical ideas accepted in completely finitary mathematical theories.

D. Hilbert [18] pointed out the advantages of such construction of arithmetics and algebra where terms like “a natural number”, “an integer”, “a rational number”, “a polynomial with integer coefficients” are defined as special strings of signs characterized by generation rules (as different from ‘abstract entities’ usually understood when corresponding areas of mathematics are build up axiomatically or set-theoretically).

The term “natural number” is understood below as a general name for words $0, 0|$, $0||, 0|||, \ldots$ which are characterized by ‘natural’ generating rules (and as abbreviations of these words we use corresponding decimal numerals). Integers and rational numbers are also defined by ‘natural’ generating rules using signs $0, |, -, /$. (For rationals one introduces a special equality relation different from the graphical equality, and all considerations use this specific equality relation as a ‘background’.) (Appendix D.)

Further examples of *constructively definable objects* (CDO) which attracted Hilbert’s attention were formulas of logical languages, derivations in logical calculi, etc.

L. Kronecker showed that it is possible to define the notion of algebraic number in such a way, that algebraic numbers are characterized as CDO and a notion of a real number is not used at all. A ‘more visualizable’ definition is given in [6]. (Appendix E.)

After Hilbert distinguished CDO as specific objects under consideration, he sketched a specific way to consider them, which he called *a finitary position* (*a finitary standpoint*). One can trace some sources of this position to some statements by Kronecker. The finitary standpoint considers CDO as ‘visualizable objects’ [18] using an idealization called abstraction of potential infinity (or potential realizability) but without abstraction of completed (actual) infinity. Here one “…uses direct contentual arguments, performed as thought experiments over visualizable objects and not depending on assumptions of an axiomatic nature”. (Appendix F.)

The latter formulation is a restriction on admissible types of mathematical arguments which is very ‘strict’ from one side but ‘fuzzy’ from the other side. It has ‘approximately orienting’ character and implicitly includes restrictions of used language constructs. Specific examples considered by Hilbert and some of his formulations allow to make precise (preserving Hilbert's main idea) the type of judgments which can be considered as meaningful from the finitary point of view. In the number theory these are judgments of the form

$$S_0 \quad \exists^+ y_1 \ldots \exists^+ y_m \forall x_1 \ldots \forall x_n R \quad (m \geqslant 0, n \geqslant 0),$$

where $\exists^+$ is the *constructive* existential quantifier (quantifier of *potential realizability*) explained by a word combination "it is possible to construct" (or "potentially realizable"), R is an algorithmically testable (decidable) $(m+n)$-ary condition and $y_1, \ldots, y_m$, $x_1, \ldots, x_n$ are variables for natural numbers. Outside the theory of natural numbers these are judgments which can be 'translated' into judgments of the above form by arithmetization, i.e. by an effective encoding into natural numbers of objects of all types in the theory considered. (Appendix G.)

Judgments of the form S_0 belong to such a 'narrow' type of judgments actually employed in mathematics (in particular in mathematical analysis) that at first sight one cannot hope to express in this language 'in a practically satisfactory way' even 'the most applicable' parts of MA.

Finitary standpoint in the monograph [19] (and before that in [18]) was demonstrated first using the example of elementary arithmetic, which includes constructively defined primitive recursive functions (PRF), and of elementary algebra. In the further parts of [19] it was tested on the material of language construction of logical and logico-mathematical languages and derivations in logical and logico-mathematical calculi. This standpoint was not proposed by Hilbert as a recommended methodology for the whole of mathematics. Moreover, after mentioning the possibility to construct foundations of the theory of algebraic numbers in the framework of finitary standpoint discovered by L. Kronecker, Hilbert and Bernays claim that in MA "…non-finitary ways of definition and non-finitary proofs are form an indispensable part of the methods of a theory".

Thus Hilbert, although he proposed finitary standpoint as a suitable foundation for investigation of properties of logical and logico-mathematical calculi (in the first place of their consistency) and as a method of construction of relatively elementary fragments of mathematics, did not expect that it might be possible to construct finitary variants of 'sufficiently rich' mathematical theories. He stressed the dominant role in mathematics of the axiomatic method based (in its logical part) on the deductive apparatus of classical logic, which is motivated by intuitive ideas of the CST.

However, our discussion of the problem of 'finitization' of MA is not concerned with some variant of 'retelling' definitions and statements of the traditional MA into a language admissible in finitary mathematics: such attempts are hopeless. We would like to construct by means of such a language a system of notions which *may be considerably different from the traditional system* but allows to develop an *apparatus* of MA to a degree 'practically sufficient' for traditional applications and (hopefully) in a form admitting additional possibilities (for example allowing to 'see' clearly the types of constructions participating in solutions of the problem under consideration, as well as the types of data for such constructions). It can happen that finitary analogs of some theorems of MA turn out to be 'finitary strengthenings' of the analogs of these theorems in a constructive MA, which was developed in the XX century on the basis of 'immediate' constructive analogs of the basic notions of the classical MA in the framework of 'weaker' requirements of 'semantical clarity' than in finitary mathematics. (This 'traditional' constructive MA uses much wider language than that of finitary mathematics.)

From this viewpoint of the goal to be achieved one can say that the opinion of Hilbert and Bernays about 'non-finitizability' of mathematical analysis was refuted by the further development of mathematics.

The main goal of this paper is to confirm this conclusion.

2. Particular theories of natural numbers and arithmetical algorithms as an 'environment of modeling' finitary theories of constructively definable objects of various types

2.1. The theory of CDO uses a standard technique of coding arbitrary words in a given alphabet, words of special types characterized by generating systems and CDO of other types by natural numbers. This technique allows to 'translate' many questions of finitary mathematics into the language of arithmetics (the theory of natural numbers) and arithmetical algorithms. This possibility is used systematically below.

The term "recursive function" is used below as a synonym of the term "partial recursive function".

2.2. From the finitary standpoint the problem of explanation of mathematical judgments about CDO is a *problem of extrapolation* to such 'almost physical objects' of the ideas about true judgments (characterized in Section 1.4 above) which were shaped by completely finitary situations. Attempts to extrapolate such semantics to the theory of natural numbers cause principal difficulties. A significant role in the understanding of the character of these difficulties was played by the scale of constructive ordinals developed by Cantor.

Principal difficulties appear already for arithmetical sentences of the following two forms S_1 and S_2:

S_1 *A recursive n-ary arithmetical function f is total.*

A symbolic translation explaining this judgment has the form

$$\forall x_1 \ldots \forall x_n ! f(x_1, \ldots, x_n).$$

Here $x_1, \ldots x_n$ are variables for natural numbers and notation $! f(x_1, \ldots, x_n)$ means: "the process of computing of the value of $f(x_1, \ldots, x_n)$ terminates".

S_2 *A total n-ary recursive function ϕ is a 0-function.*

In the symbolic notation,

$$\forall x_1 \ldots \forall x_n (\phi(x_1, \ldots, x_n) = 0).$$

A discussion of difficulties we have in mind here is presented (with a short history) in [45, Section 1]. Let us only note here that as a result of analyzing principal difficulties one has to 'resign' to the impossibility of any 'complete and final' refinement of an idea of 'convincingly motivated' extrapolation to judgments of the form S_1 and S_2 of the notion of a true sentence used in completely finitary situations. In this sense the problem of 'exact' semantics for such sentences can be called a 'deadlock'.

This motivates one to choose *particular semantics* for those sentences (as well as for the sentences which can be reduced to them by exact semantical definitions, for example judgments of the form S_0) and to 'work' with judgments true in the chosen particular semantics.

2.3. One usually uses a method of presenting particular semantics based on defining a notion of a "true judgment" by a pair of generating systems (generating grammars) Γ and Σ having the properties listed below.

(1) Objects generated by the system Γ (call them Γ-functions) are sign combinations determining (in a precise sense) some arithmetical recursive functions and composed (for example) in the way H. Curry proposed in [8] (cf. also, [35], [36, Section 2]) for PRF composed from symbols of initial functions and some operators. (For Curry's notation for PRF these are symbols for operators of regular substitution and primitive recursion.) The sets of generated objects for actually used systems of this kind usually are decidable.

(2) It is assumed that some 'sufficiently visualizable' ('contentual' in the sense of Hilbert) *justification of totality* of every Γ-function is given. (For example justification of totality of any particular PRF uses 'familiar' inductive argument for the standard ordering of natural numbers, and there is no need in some more complex form of induction.)

(3) To present 'composite' recursive operators, which are natural in the framework of a given notation (we call them Γ-operators below), one introduces sign combinations, which are obtained by using variables for Γ-functions together with original Γ-functions in the construction processes (by means of which Γ-functions are characterized), followed by binding of all functional variables occurring in the generated expressions by Church's λ-symbol.

(4) It is assumed that, based on the generating system Γ, a language (called a Γ-language below) is introduced. Formulas of this language are constructed in a standard way from natural numbers, variables for natural numbers, Γ-functions, variables for Γ-functions, Γ-operators, variables for Γ-operators and sign $=, \&, \vee^+, \forall, \exists^+$. The signs $\vee^+$ and $\exists^+$ stand here for constructive (Brouwer's) disjunction and existential quantifier, respectively. 'Classical' disjunction $\vee$, 'classical' existential quantifier $\exists$ and equivalence $\leftrightarrow$ are introduced as abbreviations:

$$(P \vee Q) \stackrel{\text{def}}{=} \neg(\neg P \& \neg Q), \quad \exists x P \stackrel{\text{def}}{=} \neg\forall x \neg P,$$

$$(P \leftrightarrow Q) \stackrel{\text{def}}{=} ((P \to Q) \& (Q \to P)).$$

One counts as *finitarily meaningful* closed formulas of the form

$$S^* \quad \exists^+\zeta_1 \ldots \exists^+\zeta_k \forall\xi_1 \ldots \forall\xi_l \Xi,$$

where Ξ is a *quantifier-free* formula and $\zeta_1,\ldots,\zeta_k$, $\xi_1,\ldots,\xi_l$ are distinct variables, each of them ranging over either natural numbers or Γ-functions or Γ-operators. A finitary justification of the formula S^* consists in producing (or describing a method of construction) of some sign combinations $Z_1,\ldots,Z_k$ in the range of variables $\zeta_1,\ldots,\zeta_k$ (respectively) plus a 'finitary admissible' justification of validity of the quantifier-free formula Ξ^+ which is obtained from Ξ by substituting words $Z_1,\ldots,Z_k$ for variables $\zeta_1,\ldots,\zeta_k$, respectively.

(5) To distinguish quantifier-free formulas admitting 'finitarily acceptable' validity justification one uses a generating system Σ. It is assumed that all objects generated by Σ (call them Σ-objects) are *quantifier-free* formulas of the Γ-language, and that a 'sufficiently visualizable' validity justification for all Σ-formulas is given. (For example, to justify the validity of every formula of the language of primitive recursive arithmetic generated by the standard logical deduction apparatus for this arithmetic one uses an 'ordinary' inductive argument.)

The specialization of the notion "true sentence" by means of presenting Γ and Σ states that exactly those $\forall$-closures of the formulas generated by the system Σ are considered to be "true" that are based on a 'sufficiently motivated' extrapolation (into theory of natural numbers and arithmetical functions) of the idea of true sentence used in completely finitary theories.

However, every such extrapolation is 'open for extensions'. Particular semantics can be 'arranged' into hierarchies according to 'difficulty' and our confidence in the two justifications mentioned in (2) and (5). Acceptability of the stages of these hierarchies as 'motivated extrapolations' of the notion of a true judgment used in completely finitary situations becomes less and less convincing when one moves away from the beginning of a hierarchy.

2.4. At the same time 'initial stages' of some hierarchies turn out to be 'sufficiently motivated' and at the same time suitable as a semantical base for development of 'rich in content' theories of finitary mathematics. From this point of view, the primitive recursive functions (PRF), the *quantifier-free* language constructed on their base (and using equations of primitive recursive terms as atomic formulas and 'boolean combinations' of such atomic formulas as formulas of the language) and quantifier-free apparatus of classical logical deduction with the rule of substitution of terms for variables, complemented by the quantifier-free rule of 'ordinary' arithmetical induction and postulates, characterizing the equality relation, initial functions and the operators used, attracted special attention. The whole complex was called "primitive recursive arithmetic" (PRA). It is described in [19] as an axiomatic system. Basic features of PRA in a non-formalized form can be found in [47]. Very interesting versions of PRA were suggested by Goodstein [12, 14] (cf. also [35]) and by Church [5].

When a formula of PRA is used as a judgment it is understood that a sequence of universal quantifiers bounding all variables occurring in the formula is 'invisibly' present in front of the formula. The traditional language of PRA is obviously 'completed' to a language including formulas of the form S^*.

Admissibility of PRA is motivated by the arguments admissible under finitary standpoint and justifying the totality of every PRF and validity of every formula derived by the deductive apparatus of PRA.

In connection with a great interest in modern mathematics to the question of possibility to solve some problems by 'as simple as possible' (in some sense) constructive means, significant attention is attracted by some subrecursive arithmetics. These are particular finitary arithmetics 'smaller' than PRA (cf. for example [30]; cf. also [38]). An example of a well-developed theory of this type is the theory of Kalmar elementary recursive functions.

It makes sense to talk of *axiomatic theories* of particular finitary arithmetics. One has in mind 'general' study of pairs Γ, Σ of generating systems satisfying the conditions given above and some additional conditions. Fixing additional conditions imposed for such a pair, one gets a theory of particular finitary arithmetics of a definite type.

In the following we do not use an axiomatic approach, but have in mind primitive recursive arithmetic as a concrete but typical (in many respects) example of a particular finitary arithmetic.

3. Approach of R.L. Goodstein to a construction of finitary versions of initial parts of mathematical analysis

3.1. The notion of a function of real variable which is central for traditional MA is 'connected' with the notion of a real number and is an instance of the notion of a map of a set into another set. The search for ways to 'constructivize' and 'finitize' MA drew attention in the first place to two problems. (a) How to define reasonably analogs of these two notions? (b) Is the introduction of an analog of a notion of real number as a basic notion unavoidable (unavoidability of the introduction of *some* real numbers for *special applications* is indisputable)?

The second question is caused by the fact that experimentally interpretable mathematical objects in applications of mathematics often turned out to be *interval functions* corresponding to some kind of ideas of 'approximately defined' value of the function considered for an 'approximately defined' value of the initial data. With respect to interval functions used in MA, 'pointwise defined' real functions primarily play the role of an instrument which allows to assign to an 'approximately defined' value of an initial data (usually a rational closed interval) some 'approximately defined' value of the function (usually in the same form).

In particular, if f is a continuous function and g is a Lebesgue integrable function then suitable interval functions Ω_1 and Ω_2 can be defined (for example) as follows:

$$\Omega_1(f,[r,s],n) \stackrel{\text{def}}{=} \left[\left\langle \inf_{[r,s]} f \right\rangle^{n-}, \left\langle \sup_{[r,s]} f \right\rangle^{n+} \right],$$

$$\Omega_2(g,[r,s],n) \stackrel{\text{def}}{=} [\langle C\rangle^{n-}, \langle C\rangle^{n+}],$$

where r and s are rational numbers such that $r<s$, n is a natural number. Expressions $\langle\alpha\rangle^{n-}$ and $\langle\alpha\rangle^{n+}$ where α is a real number denote a rational approximation of α up to 2^{-n} from below and (respectively) from above defined in some fixed way; C denotes the integral mean value of the function g over $[r,s]$. (Introduction of the variable n together with the 'variable closed interval' $[r,s]$ is not necessary, but is convenient technically.) In this part of the article it is not assumed that the function f (number α) is defined by algorithms, and speaking of methods of rational approximation of the number α we rely on ideas connected with this term in classical MA.

In the case of continuous functions, a possibility in principle to replace them by interval functions satisfying certain conditions was noticed by mathematicians belonging to the intuitionistic direction in mathematics (cf. [17]). A similar approach was developed in the theory of recursive functionals and operations. Construction of some fragments of the theory of differential and integral equations using the language of functions of domains enriched methods and content of mathematical physics (cf. [16]).

Although thinking in terms of functions of domains seems very natural from the viewpoint of applications of mathematics, one usually employs the techniques (in particular, 'pointwise defined' real functions) which determine the functions of domain only *indirectly*. The problem is that an 'obvious' definition of differentiation of an interval function leads to 'pointwise defined' functions (i.e. leads outside the realm of the functions of domains). Moreover, it is easy to present formulas for interval functions corresponding to (say) sin, ln etc., but it is very difficult to deduce even the simplest property of these functions based on their interval representations. This fixed the idea of the basic role of the traditional notion of real function and real number in MA in the mentality of the majority of mathematicians.

3.2. The first attempt at the 'constructivization' of the notion of real number was made by Weyl [50]. He fixed a certain language allowing, in particular, to represent some subsets of the set of rational numbers by formulas with one free variable for rationals. The formulas of this language defining sets satisfying the properties of upper classes of Dedekind cuts in the set of rationals are considered to be "definable" real numbers. Weyl developed (using apparatus of classical logic) a theory of definable numbers and definable functions. As it turned out later, this theory due to some of its peculiarities cannot play the role of a satisfactory base for 'constructivization' of MA. Weyl himself criticized his theory, but at the center of his criticism were not technical or logical aspects of the theory, but the following fundamental fact: "This was truly *atomistic theory of continuum* logically consistent but forced. By an epistemological analysis I tried to explicate as distinctly as possible the deep abyss that separates our mathematical constructions from the immediately experienced continuity" [51].

In fact, Weyl resumes here criticism of intuitive ideas of 'formation' of a line segment from separate points originating with ancient thinkers. He accepts Brouwer's ideas of a numerical continuum as an 'environment for free choice of sequences of rationals'. But these ideas were appreciated as 'sufficiently understandable' by only few mathematicians. After the intuitive notion of algorithm was made mathematically precise (and standard) in the form of a notion of "Turing machine" and some equivalent notions, the efforts of mathematicians who tried to construct a constructive version of MA were directed mainly to development of the theory of 'algorithmically definable' (in several senses) real numbers and of functions 'algorithmically defined' for such numbers. However, Weyl's criticism of the fundamental peculiarity of his theory is completely applicable to theories dealing with 'algorithmically definable' real numbers.

3.3. A principally new approach to the problem of construction of constructive (even finitary) variants of some initial chapters of MA was proposed by Goodstein [12, 13, 15]. He chooses as basic objects of MA 'approximatively definable' functions of some types considered as finitary analogs of functions that are continuous (or continuously differentiable) on a given rational segment, as opposed to 'pointwise defined' real function (as done in the traditional MA). These analogs are remarkable since their definition does not assume that reals are previously defined in any form. The main ideas of the approach of Goodstein are presented below with some differences of technical and terminological character.

First of all, a method for definition of some rational-valued functions of rationals (as well of some sequences of such functions) using PRF is fixed. As 'images' of rational numbers in the language of PRA one uses triples of natural numbers (a triple K, L, M 'describes' a rational number $K - L/M + 1$) and standard definitions of equality, order relations and basic operations are translated into the language of such 'descriptions'. Use of symbols playing the role of variables for rational numbers is permitted: every such symbol is treated as a notation for a triple of distinct variables for natural numbers. Use of symbols playing the role of notations for certain rational-valued functions defined for tuples of natural numbers (and hence sometimes on rational numbers) is permitted: in every such case symbol one assumes that a triple of PRF is fixed to 'explain' the employed notation for the rational-valued function. If a symbol F is explained by a triple F_1, F_2, F_3 of PRF then the expression of the form $F(H)$ is understood as a notation for the expression $F_1(H), F_2(H), F_3(H)$. Use of symbols playing the role of variables for rational-valued functions is also permitted. If V is such a symbol, then the expression $V(H)$ is understood as a notation for the expression $V_1(H), V_2(H), V_3(H)$ where V_1, V_2, V_3 are distinct variables for PRF with a suitable number of arguments.

A triple F_1, F_2, F_3 of ternary PRF is called a *rational monadic PRF* if it transforms equal rational numbers into equal rational numbers, and PRA derives a formula translation of this statement. A triple G_1, G_2, G_3 of 4-ary PRF is called a *sequence of rational* *monadic PRF* if PRA derives a formula translation of the following statement: every triple of ternary PRF obtained from G_1, G_2, G_3 by fixing a value of the first argument is a rational monadic PRF.

A variant of a finitary analog of a notion of a function uniformly continuous on a rational segment $[a,b]$ proposed by R.L. Goodstein can be defined in the following way (letters i,k,l,n below are variables for natural numbers and r,s are variables for rational numbers):

A sequence G of rational PRF is called a *recursive uniformly continuous function* in $[a,b]$ if it is possible to construct PRF h, p and q such that for all admissible values of variables the following conditions are satisfied:

$$(k \geqslant h(i) \;\&\; l \geqslant h(i)) \to (|G(k,r) - G(l,r)| < 2^{-i}),$$

$$(|r-s| < 2^{-p(i)} \;\&\; n \geqslant q(i)) \to (|G(n,r) - G(n,s)| < 2^{-i}).$$

Goodstein had in mind (but did not say explicitly) that these conditions should be supported by derivations in PRA.

Using a similar approach, Goodstein introduced finitary analogs of the notions of a function continuously differentiable in $[a,b]$ and of a plane curve [13].

The finitary analogs of several notions of traditional MA suggested by Goodstein are defined by a method 'similar' to the method of completion of metric spaces. There are, however, essential differences. The role of a sequence G of rational PRF mentioned above can be played by a sequence, where some of the PRF obtained by fixing a number of a term of this sequence are discontinuous in some rational points (and generally only become 'less and less discontinuous' when the number of a term of the sequence increases). Hence the rational PRFs which 'constitute' the sequences do not form a subspace of the space under construction. In Section 4.6 below we note another feature of the system of ideas suggested by Goodstein which forms an obstacle to an 'immediate' embedding of theorems stated on the base of this system into the quantifier-free language of PRA.

One has to conclude that Goodstein's approach in its original form is not sufficiently adapted for extension to many function spaces which play an essential role in modern MA. It needs modifications, and such modifications are 'suggested' by some results of development of the function space theory in classical mathematics.

A definite step in this direction was made by Goodstein himself. He constructed an algorithm (which can be turned into a recursive operator) which transforms any quadruple consisting of a sequence G of rational PRFs and a triple h,p,q of PRF satisfying the conditions above (characterizing G as a recursive function uniformly continuous in $[a,b]$) into a particular recursive function G^+ uniformly continuous in $[a,b]$ and equivalent to the function G in the 'natural' sense, such that all terms of the sequence G^+ are *polygonal* function with rational 'vertex coordinates'.

Goodstein stressed that his finitary analogs of notions of the traditional MA do not assume preliminary introduction of the notion of a real number in any form whatsoever. On the other hand, for a recursive uniformly continuous function G supplied with a triple h, p, q of PRF one can define in a natural way (by a 'diagonal type' construction) its values at algorithmically presented reals (to do this it is convenient to pass to G^+). Despite this possibility, functions introduced by Goodstein are objects of a special type corresponding to the following definition given by L. Euler: "A function of a variable quantity is an analytic expression combined in some way from this variable quantity and numbers or constant quantities."

The role of initial material for such a definition could have been played in particular by power series with rational coefficients considered as sequences of polynomials 'against the background' of certain ways of manipulating coefficients of such polynomials.

With time and depending on the situation the meaning of the term "analytic expression" has been changing (in accordance with accepted means of construction of symbolic expressions and the ways of their interpretation). As a result, one encounters in mathematics many different forms of functions in the sense of Euler. Suitable refinements of the quoted approximate characterization of the functions in the sense of Euler based on some modification and extrapolation of Goodstein's ideas of approximatively defined functions together with the results of development of theory of function spaces in the traditional MA, lead to introduction of notions which allow to develop a significant part of MA in the framework of finitary mathematics and advance realization of Kronecker's program directed to construction of various areas of mathematics using only 'careful' idealizations.

4. Finitary completions of elementary metric and countably metric spaces as finitary counterparts of function spaces of classical mathematics

4.1. The concepts of a metric space, a countably metric space, and the completion of a metric (or countably metric) space, introduced in the first half of the 20th century, played key roles in achieving the goal we talked about earlier. The concept of the completion of a metric space has its roots in the definition of the concept of a "real number" proposed by G. Cantor and, independently, by Ch. Méray. That definition is based on a certain construction.

The term "construction" is used here not in the sense that it has in constructive mathematics but in a more general sense. The definition in question appeals to 'almost physical' ideas of sequences of rational numbers as processes of sequential generation of certain objects; from this point of view, it is 'intuitively constructive'. The 'algorithmization' of this construction, with appropriate modifications and generalizations, opened a way towards building constructive and, moreover, finitary, versions of a number of areas of MA. (Appendix H.)

The definition of real numbers as fundamental sequences of rational numbers (viewed against the 'background' of a certain natural equality relation between such sequences), proposed by Cantor and Méray, competed with the definitions proposed by K. Weierstrass and R. Dedekind. But it was the Cantor–Méray definition that led to the quite fruitful generalization proposed by Hausdorff – the concept of a fundamental sequence of points of a given metric space. A version of that concept that is convenient 'for constructivization' can be defined as follows:

Let $\langle \mathbf{M}, \rho \rangle$ be a metric space (with the carrier $\mathbf{M}$ and metric function ρ), and let F be a mapping of the set $\mathbf{N}$ of all natural numbers into $\mathbf{M}$. We say that F is a *fundamental sequence of points* of this space if there exists a function h from $\mathbf{N}$ into $\mathbf{N}$ which is a *regulator of convergence in itself* for the sequence F of points, that is, a function satisfying the following condition: for any natural numbers i, k, l,

$$(k \geqslant h(i) \& l \geqslant h(i)) \to (\rho(F(k), F(l)) < 2^{-i}). \tag{$*$}$$

If a fundamental sequence F is 'exhibited' as an algorithm, but no regulator of convergence of this sequence in itself is given, then, even in the cases when the 'hidden' function h is actually an algorithm, the problem of finding (for instance) a natural number n such that, for all k, the distance from $F(0)$ to $F(k)$ is less than n turns out to be a 'creative' problem. This insufficiency of the information contained in the fundamental sequence F itself (which is felt particularly 'strongly' in the framework of constructive mathematics) prompts us to use, instead of the completions of metric spaces that consist of fundamental sequences of points, the completions that consist of *duplexes*, that is, pairs $\langle F, h \rangle$ that satisfy $(*)$ for all i, k, l.

The counterpart of such completion in the framework of constructive mathematical analysis is the result of restricting attention to constructive metric spaces, algorithmically presented sequences of points, and algorithmically presented regulators of convergence in itself (see, for instance, [33, 34, Section 10; 48]). In particular, if $\mathbf{M}$ is the set of rational numbers and $\rho(r,s) = |r - s|$ then duplex $\langle F, h \rangle$ is called a *real duplex*. It can be naturally viewed as an 'informationally wholesome' counterpart of the concept of a real number according to Cantor and Méray.

But the criticism of the intuitive base of set theory presented in Section 1 can be extended to a certain degree to the intuitive base of traditional ('broad') constructive mathematical analysis: some judgments admitted in the latter are outside of the language of finitary mathematics, and consequently difficult 'semantic puzzles' are found in them. In particular, one of the central concepts – the concept of a constructive function defined on the constructive continuum [22, 23] – is characterized by a condition that can be converted by the logical means of 'broad' constructive mathematics to the form

$$\forall x \neg \forall y \neg \forall z (\Theta(c, x, y, z) = 0),$$

where Θ is a primitive recursive function, c is the arithmetical code of the constructive function under consideration, and x, y, z are variables for natural numbers. This is an arithmetical version of that condition that is obtained as a result of encoding the constructive objects of various types used in the definition by natural numbers.

4.2. The choice of basic concepts appropriate for building finitary versions of a number of areas of MA is motivated primarily by specific theorems in various parts of MA that have the form

> Metric space $\langle \mathbf{M}, \rho \rangle$ has a countable everywhere dense subset.

Theorems of this form are usually viewed as statements of a very important property of the space under consideration – the property that allows us (whenever $\langle \mathbf{M}, \rho \rangle$ is complete) to introduce the 'isometric double' of that space, obtained by completing a certain countable metric space. But in traditional MA it is 'not common' to use the completion procedure in a systematic way as a tool for *defining* mathematical objects of 'complex' types (in particular, 'complex' elements of function spaces) on the basis of objects of simple types. But this method is what allows us to overcome Hilbert's 'taboo' on the creation of finitary versions of various areas of MA.

In many cases, traditional proofs of specific theorems about the existence of countable everywhere dense subsets in metric spaces are (or can be easily turned into) proofs of *more detailed* theorems that describe some essential features of the countable everywhere dense subsets 'exhibited' in the proof. Such more detailed theorems have the following form:

> In metric space $\langle \mathbf{M}, \rho \rangle$ there exists an everywhere dense subset $\mathbf{T}$ such that (1) its elements have individual representations as constructively defined objects of a certain specific type (we shall call them objects of type τ), so that every object of type τ 'represents' a certain element of $\mathbf{T}$, (2) objects of type τ form a decidable, and consequently recursively enumerable, set (one can think, essentially without loss of generality, that objects of type τ are words in an appropriate alphabet and that the decidability of the set of objects of type τ is understood as the decidability of a subset of the set of all words in this alphabet), (3) metric function ρ on the set $\mathbf{T}$ viewed as the set of objects of type τ is given as a certain algorithm, and (4) if X and Y are objects of type τ then $\rho(X, Y)$ is a *rational* (the alternate version: *algebraic*) number.

Any particular theorem of this kind 'demonstrates' a specific point of view on the metric space under consideration: elements of space $\langle \mathbf{M}, \rho \rangle$ are 'almost constructively defined objects of type τ.'

4.3. Some function spaces studied in MA, when defined in a natural way, turn out to be countably metric, that is to say, function ρ in the pair $\langle \mathbf{M}, \rho \rangle$ is defined on the set $\mathbf{N} \times \mathbf{M} \times \mathbf{M}$ (or on the set $\mathbf{N}^k \times \mathbf{M} \times \mathbf{M}$ with $k > 1$), where $\mathbf{N}$ is the set of natural numbers, so that for all particular values of the numeric argument(s) function ρ 'behaves' as a semi-metric function on $\mathbf{M}$. Function ρ is called a multi-metric function on $\mathbf{M}$. What we said above about everywhere dense subsets of some metric spaces applies to countably metric spaces as well.

In the theory of uniform spaces there is a theorem asserting that every countably metric space is uniformly equivalent to a metric space. Due to this theorem, countably metric spaces are given little attention in classical mathematics (except for countably normed spaces). However, when finitary counterparts of function spaces are defined, replacing some countably metric spaces by metric spaces in accordance with the theorem mentioned above can be counterproductive, because the metric function replacing the family of semi-metric functions may turn out to map some objects of type τ to nonalgebraic numbers. On the other hand, similar considerations suggest that some function spaces that are treated as metric spaces in their definitions that are natural from the perspective of classical MA can be usefully 'turned' into countably metric spaces. (This has to be done in practically all cases when at least one end of the interval playing the role of the domain of functions under consideration is not rational, or not algebraic.)

4.4. As examples, we will consider some function spaces of classical MA. In each example, a and b stand for rational numbers; ρ is the metric or multimetric function in $\mathbf{M}$; f and g are arbitrary points of the function space under consideration; τ is the name of a type of element of the everywhere dense subset of the function space.

- *Space of the functions uniformly continuous on* $[a,b]$.

$$\rho(f,g) = \sup_{[a,b]} |f - g|,$$ objects of the type τ are polygonal functions with rational "vertices" represented as a finite sequence of pairs of rational numbers (alternatively, objects of the type τ may be polynomials with rational coefficients).

- *Space of the functions continuous on the half-open interval* $[0,\pi)$.

Assume that the irrational number π is given (for instance) as a primitive recursive sequence of rational numbers γ such that, for all k and l, $\gamma(k) \leqslant \gamma(k+l)$ and $\gamma(k+l) - \gamma(k) < 2^{-k}$.

$$\rho(k,f,g) = \sup_{[0,\gamma(k)]} |f - g|,$$ τ is understood as in the previous example.

The multimetric function ρ under consideration can be used (in a specific way) to define the space of the functions uniformly continuous on the closed interval $[0,\pi]$. The method of using the function ρ for this purpose will be discussed later in Section 4.6.

- *Space of the functions m times uniformly differentiable on* $[a,b]$.

$$\rho(f,g) = \max_{0\leqslant n\leqslant m} \left(\sup_{[a,b]} |D^n(f) - D^n(g)| \right),$$ objects of the type τ are polynomials with rational coefficients.

If f is an element of type τ then $D^n(f)$ is understood as the *formal* nth derivative of f.

- If we replace sup in the last equality with $\int$, we shall get the metric function in *the space of the functions that have m generalized derivatives in the sense of Sobolev on* $[a,b]$.

- *Space of functions infinitely differentiable on* $(-\infty, \infty)$.

$$\rho(k,m,f,g) = \max_{0\leqslant n\leqslant m}\left(\sup_{[-k,k]} |D^n(f) - D^n(g)|\right),$$ objects of the type τ are polynomials with rational coefficients.

- *Space of the functions absolutely continuous on* $[a,b]$.

$$\rho(f,g) = \operatorname*{Var}_{[a,b]}(|f-g|) + |(f-g)(a)|,$$ τ is understood as in the first example.

- *Space of the functions whose nth power is summable on* $[a,b]$ (n is a positive integer).

$$\rho(f,g) = \left(\int_{[a,b]} |f-g|^n\right)^{1/n},$$ objects of the type τ are step functions with finitely many rational 'steps' represented as finite sequences of pairs of rational numbers (alternatively, objects of the type τ may be polynomials with rational coefficients).

- *Space of Lebesgue measurable functions on* $[a,b]$.

$$\rho(f,g) = \int_{[a,b]} \frac{|f-g|}{1+|f-g|},$$ τ is understood as in the previous example.

Alternatively, $\rho(f,g)$ can be taken to be the mean metric value of $|f-g|$ on $[a,b]$.

In transition from space of functions on the numeric continuum or its parts to spaces of functions defined on multi-dimensional numeric spaces or their parts, it is often natural to use appropriate multi-metrics (in ways 'similar' to the second and fifth examples above). As a preliminary step, one needs to introduce approximative counterparts of lines and regions of a number of types found in the corresponding parts of traditional MA. (Problems that need to be resolved here are usually technical in nature rather than fundamental.) The first step in this direction was made in [13]. See also [9] where a constructive (essentially finitary) counterpart of the concept of an analytic function is discussed.

4.5. Consider some other examples of metric spaces that illustrate the remarks made in Section 4.2.

- *Space of Lebesgue measurable subsets of the numeric continuum that have a finite measure*.

$$\rho(P,Q) = \mathrm{mes}((P\backslash Q)\cup(Q\backslash P)),$$ objects of the type τ are the unions of a finite collection of open intervals with rational ends and of rational points given as finite sequences of pairs of rational numbers and of separate rational numbers. (Appendix I.)

P and Q are arbitrary measurable subsets with a finite measure.

Important in functional analysis are also compact metric spaces, whose definition, convenient for 'constructivization', has the following form: a *compact metric space* is a well-bounded (that is, having an ε-net for every positive rational ε) and complete space. If metric space $\langle \mathbf{M}, \rho \rangle$ are objects of type τ satisfy all the conditions listed in Section 4.2 above then it turns out to be useful to define finitary counterparts of *totally bounded* subspaces of the given space in the following way.

If P and Q are finite sets of objects of type τ, we define

$$\rho^*(P, Q) = \text{"the Hausdorff distance between } P \text{ and } Q\text{"}$$

(this distance is an algebraic number). The elements of the finitary completion (see below) of the space formed by the finite sets of objects of type τ, with distances between sets measured by the metric function ρ^*, are natural finitary counterparts of totally bounded spaces of classical MA. Uniformly continuous functions on such finitary counterparts are defined in a straightforward way, 'Goodstein-style'.

In classical mathematics, the theory of functionals and operators of certain types is a key part of functional analysis. In many cases, the original definitions of these concepts were quite different in character from ideas of finitary mathematics. But, in the process of development of functional analysis, in some principal cases, it has been determined that the operators in question can be approximated in a 'sufficiently interesting' sense by 'finitarily describable' operators of the same type. In such cases, approximative definitions can be used in the corresponding parts of the theory of operators as well. Here is one example of this kind.

In classical MA, an operator is called *totally continuous* if it is continuous and the image of every bounded set has a compact closure. On the other hand, a theorem in the theory of operators asserts that, in the space of totally continuous operators defined (for instance) on the space of functions whose square is summable on an segment $[a, b]$, the finitely dimensional operators defined by finite square matrices of arbitrary size with rational elements form an everywhere dense subset (the distance between two operators being defined as the norm of their difference).

4.6. Any specific theorem about an everywhere dense subset of a metric (or countably metric) space that has the form described in Section 4.2 suggests a way to define a finitary counterpart for that space. The definition we have in mind is based on the idea of completing an appropriate 'elementary' metric (countably metric) space that consists of CDO's of a certain type τ and satisfies conditions (1)–(4). Of course, objects appended to the initially taken 'elementary' space acquire definitions in the framework of finitary mathematics. The finitary counterpart of a metric space $\langle \mathbf{M}, \rho \rangle$ from traditional mathematics that is created in this way can 'exist on its own' without appealing even to the idea of the definition used originally to characterize the elements of $\mathbf{M}$ (for instance, in the case of the space of functions that are Lebesgue integrable on $[a, b]$, without appealing to the idea of a "function that is defined on all real numbers from $[a, b]$ and is Lebesgue measurable").

In constructive mathematics we find both mathematical theories of the finitary type (with their quite 'rigid' requirements on the 'clarity' of judgments and definitions) and theories that are constructive in a 'broad' sense. Theories of the latter type include not only judgments that are clarified on the level of requirements of the finitary position but also judgments about CDO's that contain complex combinations of quantifiers and propositional connectives causing semantical problems; generally speaking, judgments of forms S_1 and S_2 from Section 2.2 are viewed as 'understandable in a straightforward way'. Theories of the finitary type belong to this second type as well, because 'broad' constructivism does not claim to 'refute' theories of the finitary direction. In the version of mathematical analysis that was developed in the framework of "broad" constructivism (see [22, 23, 33, 34, 48, 49, 53]), the objects appended to the 'elementary' metric space when it is constructively completed are duplexes of the form $\langle F, h\rangle$, where F and h are algorithms that are total in a sense that is 'understandable in a straightforward way' and satisfy condition $(*)$ that is 'understandable in a straight forward way' as well. In some cases, the duplexes are appended only when they satisfy some additional condition (see [9]).

In the process of completing a countably metric space, when ρ 'depends' (for instance) on a single numerical parameter, the F and h that form together an element of the completion are total algorithms, h is defined on pairs of natural numbers, and the condition

$$(k \geqslant h(m,i) \& l \geqslant h(m,i)) \to (\rho(m,F(k),F(l)) < 2^{-i}) \qquad (**)$$

is assumed to be satisfied for all natural numbers m, i, k, l.

Let W be a total algorithm that maps every pair of natural numbers to an element of the space that is being completed, and let H be a total ternary arithmetical algorithm. The pair $\langle W, H\rangle$ is called a sequence of points of the completion if, for all n, m, i, k, l,

$$(k \geqslant H(n,m,i) \& l \geqslant H(n,m,i)) \to (\rho(m,W(n,k),W(n,l)) < 2^{-i}). \qquad (***)$$

For such sequences, the notion of a regulator of convergence in itself is introduced in a straightforward way, and the triples $\langle W, H, R\rangle$ such that R is a regulator of convergence in itself for $\langle W, H\rangle$ are called fundamental sequences of points of the completion of the 'elementary' countably metric space under consideration. Finally, a *theorem about the completeness of the completion*, in the operator form, is stated and proved: the operator Lim is constructed that maps every triple of the form described above to an element of the completion that is the limit of this fundamental sequence in a natural sense.

Including the regulators of convergence in itself, and regulators of other types, in newly introduced objects of approximative nature as their 'parts' allows us, both in constructive MA and in finitary MA, to formulate theories of appropriate objects in the *operator* version. In a number of ways, this version is more attractive than the *predicate* version. Several advantages are demonstrated, in particular, by the operator presentation of some parts of constructive MA in [33, 34]. For details on the 'relationships' between the two versions, see [27, 28]. Goodstein constructs his version of MA in the predicate form. In this, he follows the method of presentation common in classical mathematics. He does not include the regulators (of various types) that appear in his definitions as 'parts' in the objects that are being defined, and that is what prevented him from realizing his approach to MA *directly* in the framework of PRA.

In connection with what we said above, let us consider the space of the functions uniformly continuous on the closed interval $[0, \pi]$. As mentioned in Section 4.4, one may regard the space of the functions continuous on the half-open interval $[0, \pi)$ as a result of completion of the 'elementary' space, which consists of the objects of the type τ (in our case, these objects are polygonal functions represented as finite sequences of pairs of rationals or polynomials with rational coefficients), by means of the specific multimetric function ρ. In the passage to the constructive mathematics, one keeps in mind the constructive completion of the type characterized above.

In order to define in the same 'style' the space of the constructive functions uniformly continuous on the closed interval $[0, \pi]$, it is sufficient to select those elements, which have regulators of equicontinuity. We say that an arithmetical algorithm ρ is a *regulator of equicontinuity* of a duplex $\langle F, h \rangle$, which belongs to the above-mentioned completion, if for any i and k and for any rational numbers r and s the following condition holds:

$$(0 \leqslant r < s \leqslant \gamma(k) \,\&\, s - r < 2^{-q(i)}) \rightarrow (|\{F(k)\}(r) - \{F(k)\}(s)| < 2^{-i}).$$

Here $\{F(k)\}(r)$ denotes the value at r of that rational-valued function of a rational argument, which is determined uniquely by the object $F(k)$ of the type τ.

In order that the objects, considered as constructive analog of uniformly continuous on $[0, \pi]$ functions, possess 'sufficiently complete' information, it is advisable to consider as these objects the triples $\langle F, h, q \rangle$ such that condition $(**)$ holds and q is a regulator of equicontinuity of the duplex $\langle F, h \rangle$.

For the closed interval $[0, \pi]$, constructive analog of the remaining functional spaces introduced in Section 4.4 for the closed interval $[a, b]$ with rational endpoints can be defined in a similar way using 'regulators' of suitable types.

Transition from the way of completing an 'elementary' space described above to a *finitary* completion is predicated on the choice of some particular finitary mathematics as a base for constructing specific mathematical theories. This step has been discussed in Section 2. For technical reasons, it is useful to encode CDO's of various types that appear in the theory under consideration by natural numbers, and to use arithmetical recursive functions as standard algorithms. With this approach, PRA turns out to be a convenient (and 'practically sufficient' in a wide variety of cases) particular finitary arithmetic, and in the rest of this presentation we will assume, as an example, that PRA is chosen to play this role.

For *finitary completion* of an 'elementary' metric (or countably metric) space, one has to use arithmetical codes of CDO's of several types. We assume certain fixed methods for coding of objects of type τ, rational and algebraic numbers by natural numbers, and we fix on this base a method of 'translating' into the language of PRA the algorithms, concepts and relations, which are considered in connection with formulas of type $(*)$ and $(**)$. Using this method, for any formula Δ we can construct in a natural way its *arithmetical counterpart* $\langle\Delta\rangle^{\mathrm{ar}}$.

In the finitary completion, the appended objects are duplexes of the form $\langle F^*, h\rangle$ such that F^* and h are PRFs, every value of F^* is the code of an object of type τ (F will stand for the algorithm that constructs that object), and formula $\langle D\rangle^{\mathrm{ar}}$, where D denotes formula of type $(*)$ (or, respectively, formula $(**)$), is derivable in PRA. The set of duplexes of this form is recursively enumerable. Now, if we extend the duplex $\langle F^*, h\rangle$ by appending the arithmetical code $\langle\Pi\rangle^{\mathrm{cod}}$ of some derivation Π of $\langle D\rangle^{\mathrm{ar}}$ in PRA, we will get a triplex $\langle F^*, h, \langle\Pi\rangle^{\mathrm{cod}}\rangle$ – an object that contains 'richer' information than $\langle F^*, h\rangle$. Such triplexes form a decidable subset of the set of all words in the alphabet of the language.

The concepts of a sequence of points of the completion and of a fundamental sequence of points of the completion are modified in a similar way. Finally, in the operator form, one can state and prove a *theorem on the finitary completeness of the finitary completion.*

4.7. Some concepts and theorems of classical MA that deserve attention because of their 'intuitive contents' do not find their expression in the framework of the system of concepts outlined above. But the situation changes when 'quasi-fundamental' sequences of points of the metric space are introduced. The idea of such sequences goes back to a certain property of monotone, bounded sequences of rational numbers observed in [2, p. 109]. This idea, developed in [21], serves as the basis of the *theory of fillings* of constructive metric spaces, and of constructive functions on fillings. In [21], all considerations are conducted in the framework of 'broad' constructive MA. The finitary counterpart of the notion of a totally bounded space mentioned in Section 4.5 above and the discussion of finitary completions in Section 4.6 form an appropriate foundation for a finitary version of the theory of fillings.

The discussion above concentrated on finitary counterparts of the metric and countably metric spaces that have recursively enumerable (countable in the constructive sense) everywhere dense subsets. The space of generalized functions in the sense of Sobolev and Schwarz provides an example of a topological space that does not have such a subset. But, for this space, even in the framework of set-theoretic mathematics, the basic idea of Cantor and Méray showed a way to create its version that is much more 'tangible' than its initial version. The modifications of this idea in [25, 26, 20] adapted it to the new situation and made it possible to define an approximative version of the concept of a generalized function. In [10, 11] this version is adapted to the system of concepts of 'broad' constructive MA, and in [44] the transition to the finitary counterpart of the concept of a generalized function is made. Polynomials with rational coefficients are used there as basic 'building blocks'.

4.8. In combinatorial (algebraic) topology, there is a system of concepts oriented towards approximating (usually not in the metrical sense) various topological objects by objects that have 'simple structure' (and often finitary representations), and these concepts 'work' quite well. These concepts have been known for a long time – essentially, beginning with the groundbreaking work by P.S. Aleksandrov on projective simplicial spectra [1]. Finitary versions of theories developed in this framework can be usually developed with 'little effort'. What is required is basically the replacement of the concept of a mapping (in particular, the concept of a sequence of objects) in the sense of CST by the concept of an algorithm of an appropriate type.

4.9. Let us go back now to the contents of Section 3.1 and discuss the use of approximately defined functions of some types to specify interval functions indirectly. In the finitary version of mathematical analysis, interval functions are defined first for elements of the 'elementary' space whose finitary completion produces the function space we are interested in. For instance, if the role of 'elementary' objects is played by polygonal functions with rational 'vertex coordinates' (or polynomials with rational coefficients), and they are treated as the foundation of the finitary space of the functions uniformly continuous on $[a,b]$, and if f is such as 'elementary' object then $\Omega_1(f,[r,s],n)$ (see Section 3.1) is an algorithmically presented rational interval. If, on the other hand, this role is played by step functions with finitely many rational 'steps' in $[a,b]$ (or polynomials with rational coefficients), and they are treated as the foundation of the finitary space of the functions that are summable on $[a,b]$, and if g is an 'elementary' object of that type, then the same can be said about $\Omega_2(g,[r,s],n)$. (Appendix J.)

The extrapolation of the interval function Ω_1 to elements of the finitary completion (we will denote it by $\Omega_1^\wedge$) is conducted by the 'diagonal' method. If $\langle F,h\rangle$ is a duplex that it is an element of the finitary completion then

$$\Omega_1^\wedge(\langle F,h\rangle,[r,s],n) \stackrel{\text{def}}{=} \Omega_1(F(h(n)),[r,s],n).$$

It is easy to estimate the 'measure of difference' between the rational intervals obtained in this way, depending on n and on the 'closeness' of r and s. The extrapolation $\Omega_2^\wedge$ of the interval function Ω_2 to the elements of the second completion is defined in literally the same way, and what was said above about an estimate for $\Omega_1^\wedge$ applies to $\Omega_2^\wedge$ as well.

Specifications of interval functions on the basis of approximatively defined functions, as above, are indirect specifications. But their essential feature is that they do not appeal to the concept of a real number in any form whatsoever.

5. On the theorems of finitary mathematics that have the form of majorants of theorems of 'broad' constructive mathematics

5.1. Theorems of 'broad' constructive mathematics are stated in a language that is 'richer' than the language of finitary mathematics. For instance, a judgment of the form

$$\forall x(\forall y(\phi(x, y) = 0) \to \forall z(\psi(x,z) = 0)) \qquad (****)$$

where ϕ and ψ are PRFs, and x, y, z are variables for natural numbers, is beyond the range of the language of finitary mathematics. What we said in Section 2.2 about judgments of the form S_2 prevents us from understanding the sign "$\to$" in this formula as a Boolean function.

The question of the semantics of judgments stated in the language of 'broad' constructive mathematics has a long history. A brief survey of that history can be found in [39]. One of the approaches proposed in [36] (and briefly presented in [39]) is suggested by a large variety of examples of theorems of 'broad' constructive mathematics for which one could find finitary provable majorants (that is, 'strengthenings' from the perspective of the deductive apparatus of 'broad' constructive mathematics) that have the form S^* (Section 2.3) or can be converted to that form by means of arithmetization. Such examples suggested a hierarchy of methods for constructing the majorants of arbitrary arithmetical judgments that have the form S^*. There is no reason to claim, of course, that any arithmetical judgment or its negation has a majorant in these hierarchies that is true in an 'acceptable' particular finitary semantics. But an approach to extrapolating ideas about the truth of judgments to the 'broad' arithmetical language does emerge from these considerations.

This approach enhances our understanding of properties of the languages and of the logical apparatus used in constructive mathematics, but, generally, it does not give much help in our search for the finitary versions of theorems of constructive mathematics that have an 'interesting content'. The problem is that the general methods for constructing majorants mentioned above assume the preliminary step of converting the given judgment to a special form that has no occurrences of implication and, on the whole, 'drastically' changes the given judgment.

But in many cases one can construct, on the basis of specific considerations, finitary majorants that are 'relatively close' in their logical structure to the initially given judgments and, on the other hand, are true in a certain particular semantics that is located in an 'initial part' of the hierarchy of particular semantics (for instance, in PRA). Special methods for constructing majorants that are fruitful in this respect vary from one case to another (which leads, one can say, to a library of such methods), and it turns out that some theorems of finitary mathematics can be usefully stated in the form of suitable theorems of 'broad' constructive mathematics along with an indication of a specific method for constructing majorants. When such a specific method is fixed, some theorems of 'broad' constructive mathematics can play the role of convenient statements of certain theorems of finitary mathematics. Then some fragments of the deductive logical apparatus of 'broad' constructive mathematics turn out to be admissible (possibly under some limitations) from the point of view of this understanding of the role of the theorems mentioned above.

5.2. As an illustration, consider two examples. For a judgment of form $(****)$, the role of a finitary majorant with ‘interesting content’ can be played, for instance, by the formula

$$\exists h \forall x \forall z (\forall y_{\leqslant h(x,z)} (\phi(x, y) = 0) \rightarrow (\psi(x,z) = 0)),$$

and also by the formula

$$\exists h' \forall x \forall z (\phi(x, h'(x,z)) = 0) \rightarrow (\psi(x,z) = 0)),$$

where h and h' are variables for 2-place PRFs and x, y, z are variables for natural numbers.

In Goodstein’s work on recursive MA we find theorems reducible to that of the form $(****)$ and of more complex forms, but he never describes precisely the kinds of inferences that he uses (intuitively, in fact), which, inevitably, go beyond the PRA (primarily because of the logical form of the judgments that appear in his reasoning). The analysis of the ‘logical situation’ in these papers can be found in [35]. It turns out that in a number of cases Goodstein actually justifies majorants with ‘interesting content’ for some judgments that are theorems of constructive, in the ‘broad’ sense, MA.

As the second example, consider, in the framework of classical mathematics, the judgment: “Every monotone nonincreasing PRF has a point of stabilization”. In the logico-arithmetical language it has the form

$$\forall f (\forall m (f(m+1) \leqslant f(m)) \rightarrow \exists n \forall k (k \geqslant n \rightarrow f(k) = f(n)))$$

where f is a variable for one-place PRFs, $\exists$ is the ‘classical’ existential quantifier (see Section 2.3), and m, n, k are variables for natural numbers.

This judgment has the logical structure that is much more complex than $(****)$. But from the ‘natural’ reasoning used in classical mathematics to justify this theorem one can ‘extract’ a majorant of this judgment with an interesting ‘visual meaning’ that has a finitary justification (cf. [40, 41]).

Appendix A

There was also another line of development of foundational ideas that played a major role in the formation of the ‘appearance’ of mathematics. It was not (directly) based on any analysis of the idea of ‘infinity’, and it has led to the development of the *axiomatic method* of building mathematical theories as an independent mathematical ideology. Its roots are found in Euclid’s *Elements*. This line is clearly described in the following excerpts from [3] (Chapter “Foundations of Mathematics. Logic. Set Theory”):

> “Mathematicians were always convinced that they prove ‘truths’ or ‘true propositions’. $\langle \ldots \rangle$ The traditional view of mathematical truth goes back to the time of Renaissance. In this view, there is no significant difference between the objects that the mathematician deals with and the objects studied in natural sciences. Both

were considered intelligible, mastered by man by means of intuition and reasoning. ⟨...⟩ From antiquity until the 19th century, there had been complete unanimity regarding the objects that are basic for mathematicians; ⟨...⟩ those are numbers, quantities and figures. ⟨...⟩ No matter how much philosophical ideas about mathematical objects developed by mathematicians and philosophers differed from each other in details, there was at least one thing that they had in common: those objects are *given* to us, and it is not in our power to assign arbitrary properties to them, just as the physicist does not have the power to change a law of nature. ⟨...⟩ The first blow to these classical concepts came from the development of non-Euclidean hyperbolic geometry by Gauss, Lobachevsky and Bolyai at the beginning of the century. ⟨...⟩ Gauss and Lobachevsky believed that disagreements between various possible geometries can be resolved by experience. ⟨...⟩ This view was shared by Riemann. ⟨...⟩ But this problem is clearly outside of the realm of mathematics. And it seems that not one of the authors mentioned above had any doubt that, even if some 'geometry' does not agree with experimental data, its theorems still remain 'mathematical truths'. ⟨...⟩ Finally, it becomes clear to mathematicians that ⟨...⟩ in mathematics, it is completely legal to reason about objects that have no sensory 'interpretation'. From that time on, wide use of the axiomatic method becomes common. ⟨...⟩ In other words, mathematics appears to be essentially the study of objects about which one knows nothing except for certain properties describing these objects – the properties taken as the axioms the theory is based on".

Georg Cantor 'sharpens' the idea expressed in the last sentence by saying: "...mathematics is completely independent in its development, and its concepts are bound only to be noncontradictory and to be related to concepts introduced earlier by precise definitions" [4]. In this connection, one might remember Hermann Weyl's 'correction' [52]:

"The constructions of the mathematical mind are at the same time free and necessary. The individual mathematician fells free to define his notions and to set up his axioms as he pleases. But the question is, will he get his fellow-mathematicians interested in the constructs of his imagination. We cannot help feeling that certain mathematical structures which have evolved through the combined efforts of the mathematical community bear the stamp of a necessity not affected by the accidents of their historical birth".

In its modern form, the axiomatic method of building mathematical theories includes describing a theory as a formal deductive system (which is a special case of the 'purely symbolic' concept of a *calculus*) and poses the problem of justifying the consistency of this system, but (generally) it does not require any motivation for the choice of specific axioms and logical inference rules included in the theory under consideration.

Broad use of the axiomatic method in mathematics was accompanied by 'replacing' the concept of a "true judgment" with the concept of a "judgment derivable from certain initially given judgments by certain inference rules". This 'liberation' from difficult semantical questions made the axiomatic method an attractive tool for achieving some kind of 'clarity'. This clarity is achieved at the price of eliminating from the 'field of view' any relationship (that may include, generally, extrapolations and idealizations) between judgments of the language used in a given theory on the one hand, and conjectures about the results of experiments expected in certain situations, on the other.

But the mathematical theories that are of interest in real 'mathematical life' usually have definitions that are accompanied by certain motivations. What is valued particularly highly (at least from the point of view of applications of mathematics) is a 'delineation' of at least one 'realization' of the axiomatic theory that 'traces' how it emerged in the course of modelling real-life situations of some kind using certain (preferably 'careful' or at least 'not too fantastic') extrapolations and idealizations.

Motivations that are offered 'inside' mathematics often have the nature of an interpretation of the axiomatic theory under consideration in some other axiomatic theory which has already been given an acceptable, from a certain point of view, motivation. On this path, sometimes one may be able to find an interpretation in the framework of the theory of finite sets, and sometimes in the framework of finitary mathematics, and that boils down to providing a motivation of the kind described earlier.

But interpretations (and sequences of interpretations) of axiomatic theories in other axiomatic theories (if we think of those actually studied in modern mathematics) lead eventually, in most cases, to Cantor's theory of *infinite* sets, or to 'large' fragments of that theory (for instance, due to the use of the logical deductive apparatus of classical logic whose acceptability is motivated using the idea of 'completed infinity').

Thus the 'clarity' achieved by representing mathematical theories in the axiomatic form is usually technical in nature, in a manner of speaking. The axiomatic formulation of a mathematical theory creates favorable conditions for the development of various technical aspects of that theory, which makes it a valuable tool. But an axiomatic theory that is not accompanied by a motivation is essentially (from the perspective of a natural scientist) a complex of hypothetical considerations to be 'kept in store', 'just in case'.

The history of science knows a number of cases when 'unmotivated' axiomatic theories were required as appropriate theoretical models of certain real-life situations. Non-Euclidean geometries provide a particularly good example. But the dependence of both Euclidean geometry and traditional non-Euclidean geometries (presented, for example, as appropriate 'analytical geometries') on CST is 'insignificant': 'small' changes can turn them into theories of finitary mathematics.

Appendix B

Cantor illustrates the meaning that he gives to the word "set" by saying, "A set is a collection of certain well distinguished objects of our intuition or thought taken as a whole" [4]. These explanations appeal to the 'world of images', but they involve no ‘direct’ comparison of the idea of “infinite collections of simultaneously existing objects” with any kind of results of the ‘experimental conquering’ of nature (on any level of detalization, and with any ‘extent’ in time and space), and no analysis of the nature of the extrapolations and idealizations that lead from experimental data to images of this kind. Within this approach, the doubts and questions raised by ancient thinkers do not arise, except for those related to paradoxes ‘on the level of pure thought’.

Cantor’s ‘naive’ set theory, at early stages of its development, brought mathematicians a ‘surprise’ in the form of examples of assertions that can be both proved and refuted in the framework of the theory. Soon, however, the theory was ‘corrected’ in several ways (the differences between these ways being essentially details, technical in nature), and the proposed versions have been presented as axiomatic theories. For a long time, mathematicians have been convinced that these axiomatic theories are not going to bring new ‘surprises’ similar to the one mentioned above.

But this conviction is not capable of resolving ‘hard’ questions of a very different kind – questions about the acceptability of the basic intuitions that a person actually appeals to when set theory is applied to problems having to do with experimental data of some kind. Such processes sometimes remain unnoticed, because they are usually indirect and occur, for instance, as part of the construction of a mathematical model of some real-world situation in modern theory of differential equations, or representations of groups, or random processes, and so forth.

Attention to questions of this kind varied with time. In the twentieth century, set theory enjoyed great success, in the sense that practically every area of mathematics that had come into existence by the middle of the century ‘lent’ itself to presentation in the language of that theory combined with the modern apparatus of classical logic. (That was only true, actually, as long as one refrained from discussing problems typical for the emerging constructive direction in mathematics.) Moreover, the system of concepts offered by CST helped in the creation and development of some new areas of mathematics (for instance, the theory of function spaces and operators). The codification of the system of concepts of mathematical analysis based on CST has resolved uncertainties related to its repertory of formal procedures (related, for instance, to operations with infinite series and improper integrals), and many mathematicians came to the conclusion that the ‘meaning’ of all components of this repertory has been made sufficiently clear. (Indeed, every mathematical procedure is interpreted as a set of ordered pairs whose members are sets also.) The demonstration of the “superhuman power of formal procedures” (see Section 1.2) was perceived by many mathematicians as an a posteriori justification of the acceptability of CST as a whole, including *all* abstractions used in CST. From this perspective, the very problem of characterizing and evaluating ‘how fantastic’ are the abstractions involved in the intuitive base of CST appears irrelevant.

At the same time, David Hilbert [18] discussed the realizability of the basic concepts of CST on macro-, micro- and mega-levels of detalization and ‘extent’ in the space-time, and he summarized his analysis as follows:

> "We saw earlier that the infinite is not to be found anywhere in reality, no matter what experiences and observations or what kind of science we may adduce. Could it be, then, that thinking about objects is so unlike the events involving objects and that it proceeds so differently, so apart from all reality?"

Appendix C

In applications of mathematics, when problems of this kind are being considered, one often uses theoretical models formed from the 'building materials' provided by traditional MA. The latter systematically uses the abstract concepts of set theory – concepts that are often very remote from 'specific informations'. Constructive direction in mathematics creates preconditions for search for 'purely informational' models of the fragments of the 'world of experimental data' that are being studied. We are talking here about the theoretical models in which the objects under consideration (real or imaginary) and connections between them are *individually represented* with a satisfactory degree of detail and precision by 'specific informations' (against the 'background' of some equality relation expressing the interchangeability of these 'informations' from a certain point of view). We prefer the models that treat 'specific informations' as symbolic constructions, *take into account this feature to a sufficient degree*, *do not use the idealizations that can be avoided*, and do not 'surround' them by any kind of 'ideal objects' *that have no individual representations by symbolic constructions.*

Appendix D

Key ideas related to defining mathematical concepts in the framework of the 'finitary stand' are presented in [19] as follows:

> "…in the areas of elementary arithmetic and algebra, orientation towards direct informal reasoning without assumptions of axiomatic nature is practiced in a particularly pure form. Specific for this attitude is the view of reasoning as a *mental experiment* with objects that are assumed to be *explicitly specified.* ⟨…⟩ In arithmetic, we have an initially given object and also a generating operation. Both will need to be fixed by some visual means. ⟨…⟩ Let's take the digit 1 to be the initial object, and appending 1 to be the generating operation. The objects that we will get if we begin with 1 and apply the generating operation ⟨…⟩ are the results of specific completed *constructions*."

It turned out that the idea of this constructive definition of the concept of a positive integer can be extended (in a generalized form) to many concepts from various areas of mathematics. As a result, a general theory was created (primarily in [29]) – a theory of *generating systems* or *grammars*. On the basis of this theory, the intuitive idea of a constructively defined object was standardized (and consequently made precise), with a convincing motivation.

Appendix E

In [19], Leopold Kronecker is mentioned as the founder of the 'finitary stand' in mathematics. He proposed to accept the concept of a natural number as 'intuitively clear' and then to view a mathematical concept as 'acceptable' only if it is 'expressible' in some way in terms of the concept of a natural number. (The 'expressibility' that Kronecker had in mind can be probably described in modern mathematical language as the existence of "an effective encoding of the objects in questions by natural numbers".) Appropriate in this respect, are, for instance, the concepts of an "integer", a "rational number", a "polynomial with integer coefficients", a "square matrix with rational elements".

Kronecker has arrived at a fundamental result that could serve as 'motivation' for a program of this kind: he 'effectively generated' the decomposition field for polynomials with integer coefficients into linear factors without appealing to the general concept of a real number in any form. In this way he established that the concept of an algebraic number in its traditional form can be replaced by a concept from constructive mathematics that can 'play the same role'. The elements of that field are constructively defined objects that can be encoded by natural numbers in a relatively simple way (their arithmetical codes are algorithmically recognizable among arbitrary natural numbers) against the 'background' of a certain algorithmically decidable equality relation. This result plays a crucial role in modern work on the development of finitary versions of various theories of classical mathematics, but, taken by itself, it is absolutely insufficient for solving the problems arising on this path. It did not allow Kronecker to make significant progress towards his goal.

A constructive definition of the notion of an algebraic number that is 'more tangible' than Kronecker's was proposed in [6]. According to that definition, a real algebraic number is any triple

$$r,\ s,\ P$$

(with constructively defined terms r, s, and P) where r and s are rational numbers such that $r \leqslant s$, and P is a polynomial with integer coefficients that satisfies the algorithmical test for the existence and uniqueness of a root in the closed interval $[r, s]$ that is stated in classical mathematics as a consequence of Sturm's theorem on the number of roots of a polynomial in a given interval. Any ordered pair of real algebraic numbers is called an algebraic number. For algebraic numbers defined as described above, an algorithmically decidable equality relation is defined in an obvious way, and all considerations related to algebraic numbers are conducted against the 'background' of this equality relation.

Appendix F

The term "abstraction of potential realizability" was introduced into mathematical literature by A.A. Markov. In fact, he uses this term as a name for a certain special case of what was called "potential infinity" in ancient mathematics – the case of processes of constructing combinations of symbols, discrete in time and space, by humans or mechanisms that follow some clearly specified *construction rules* for forming new combinations of symbols from combinations produced earlier (such as words in a given alphabet). The idealization that he associated with this term "…consists in abstracting from the practical limits of our capabilities in space, time and material in the process of realizing words" [23]. In [24], he observes:

> " …our capabilities are truly limited, and there is no reason to believe that it will be always possible to overcome the obstacles caused by these limitations. Just the opposite, modern physics and cosmology seem to provide evidence that such obstacles are unavoidable in principle. ⟨…⟩ Abstraction of potential realizability, like any other abstraction, brings with it an element of imagination wherever it is applied. ⟨…⟩ The difference between 'classicists' and 'constructivists' is that they use different abstractions, that is, imagine in different ways".

There is a good reason why Markov mentions physics and cosmology here. As a matter of fact, when a mathematician begins with his experience of operating with practically realizable processes of generating combinations of symbols and then goes on to theoretical ideas about 'very remote steps' of processes of a certain kind and about their expected results, he mentally abstracts from his practical limitations and, moreover, uses (without 'announcing' this) a whole 'gamut' of (generally) fantastic extrapolations. These extrapolations involve ideas about 'very large' extents of space and time, about the simultaneous existence of letters in an 'already constructed' word that are 'very far removed' from each other, about topological properties of 'very long' chains of elementary objects (for instance, we 'convince ourselves' that the process of generating words will not lead to self-intersections), and so on, and so forth.

Apparently, the mentality of a mathematician, who includes in the set of notions in use the abstraction of potential realizability, is formed usually in such a way, that the aforementioned 'gamut' of extrapolations comes from some parts of Euclidean geometry and classical mechanics. These disciplines allow the intuition to be 'based' on suitable theorems about 'arbitrary large rectangles' and combinations of such figures. At the same time one keeps in mind Newton's ideas on simultaneous events, on translations of figures as hard bodies and so on.

The understanding of the 'vagueness' of such extrapolations in theoretical considerations related to expected results of 'very long' constructive processes has led to the development of techniques directed (when the construction has an 'arbitrarily selected' number n as a part of its input) towards establishing an upper or lower bound of complexity of constructions (in a given sense) by means of 'estimating' functions from a certain collection that usually includes linear functions, polynomials, exponential and hyper-exponential functions, and so forth. Such estimates allow us to get some idea of the ‘limits of applicability’ of the theoretical models using the constructions in question.

Constructive mathematics – and this is special about it – finds its ‘support’ in the idea that ‘many’ CDOs (those generated by the processes whose duration is ‘not too large’) “…are *given* to us, and it is not in our power to assign arbitrary properties to them, just as the physicist does not have the power to change a law of nature” (see Appendix A). (It is necessary to keep in mind, of course, that the ‘elementary’ symbols that CDOs consist of are perceived as objects with an amount of detail that is not greater than what is needed to recognize their equality). This is different, for instance, in geometry. Intuitive ideas that have to do with even the simplest geometric objects (points, segments and straight lines) give reasons to discuss the ‘degree of imagination’ involved in these ideas and even to seek alternatives to them. In the case of CDOs, such reason are given only by ‘very long’ generating processes and their expected results. In this sense, we can talk about CDOs as ‘almost physical’ objects.

Appendix G

Constructive existential quantifier and constructive disjunction became part of mathematical practice when L.E.J. Brouwer declared his point of view on mathematics as the mental activity that creates mental constructions and analyzes them. The word “construction” is understood by Brouwer in a very broad sense – so broad that, besides the combinations of symbols characterized as potentially possible results of constructive processes consisting of steps of some fixed types, it includes also potentially infinite processes of sequential *free* choices of objects of these types. But, when speaking about constructively defined objects in the ‘narrow’ sense, such as natural numbers, Brouwer insisted (in particular) on constructive understanding of judgments about the existence of a natural number satisfying a given condition (Kronecker insisted on this too) and on the constructive understanding of disjunction. On the other hand, Brouwer and his followers used the language that allows *arbitrary* combinations of conjunction, implication, negation and universal quantifier. The use of such a language was viewed as intuitively acceptable (although without stating a clear semantics). But in this language one can express both ‘classical’ disjunction $\vee$ and ‘classical’ existential quantifier $\exists$. This fact makes it necessary to introduce new symbols for constructive disjunction and constructive existential quantifier, for instance, $\vee^+$ and $\exists^+$.

Constructive disjunction $\vee^+$ can be viewed as the defined logical connective characterized by the definition

$$(P \vee^+ Q) \overset{\text{def}}{=} \exists^+ n((1 \leqslant n \leqslant 2) \,\&\, (n = 1 \to P) \,\&\, (n = 2 \to Q)),$$

where n is a variable for natural numbers that has no free occurrences in P, Q.

Appendix H

Finitary versions of mathematical theories (and, in some respects, versions constructive in the broad sense as well) continue the critical analysis of Cantor's set theory in a 'positive' way: they are proposed as alternatives to the versions using set theory. Such alternatives provide certain answers to the question: Is it possible to build mathematical theories without using the set-theoretic 'mentality', and what are the advantages of such an approach? There is a historical paradox in the fact that Cantor was the author of an idea that (in combination with other ideas) provided a tool for creating alternatives to the 'mathematical world view' that originated in his own work.

Appendix I

Vitali's theorem starting that the set of objects of this type is everywhere dense in the space of measurable sets of finite measure allows us, even in the framework of classical mathematics, to simplify the theory of measurable sets in a radical way, 'without any losses'. Indeed, in this theory the equality of objects is defined in such a way that it makes no sense to talk about a set 'consisting' of separate points, and the usual definition of the concept of a measurable set by 'inscribing' arbitrary closed sets and 'circumscribing' arbitrary open sets makes essentially no sense.

Appendix J

If the finitary space of uniformly continuously differentiable functions on $[a,b]$ is defined using polynomials with rational coefficients then, instead of the interval function Ω_1, one can use 'cruder' but 'more easily computable' interval functions, such as, for instance, the function $\Omega_1^{\sim}$ defined by

$$\Omega_1^{\sim}(f,[r,s],n) \stackrel{\text{def}}{=} \left[f\left(\frac{r+s}{2}\right) - c_n \cdot \frac{s-r}{2}, \quad f\left(\frac{r+s}{2}\right) + c_n \cdot \frac{s-r}{2} \right],$$

where

$$c_n = \left\langle \max_{[a,b]} |D(f)| \right\rangle^{n+}.$$

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