Half-duplex communication complexity

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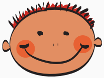
Communication models

Classical communication model

Introduced by Andrew Yao in 1979.

Alice

Bob



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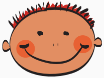

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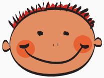

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Alice and Bob want to compute $f(x, y)$.

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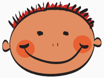
Alice $\xrightarrow[b_1]{}$ Bob



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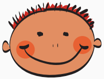
$\xrightarrow[b_1]{\hspace{4cm}}$

$\xrightarrow[b_2]{\hspace{4cm}}$

Bob



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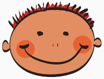
$\xrightarrow{b_1}$

$\xrightarrow{b_2}$

$\xleftarrow{b_3}$

Bob



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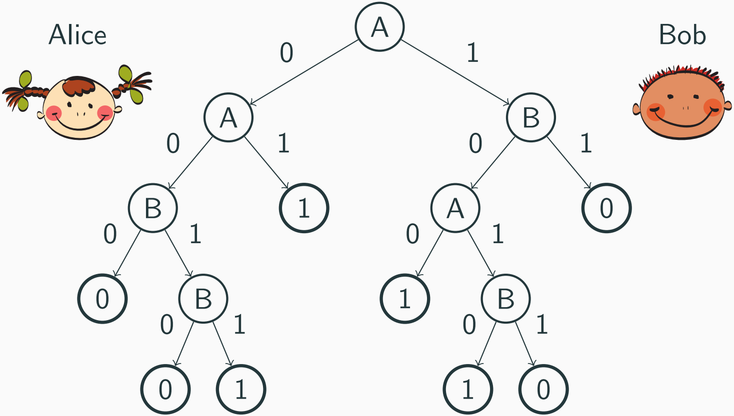
Communication protocol



Communication complexity of f is a minimal depth of a protocol solving f, denoted $D(f)$.

Half-duplex communication model

Players talk over half-duplex channel (e.g., “wakie-talkie”).

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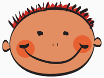

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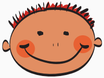

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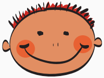

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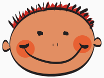

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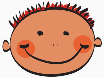
$\xleftarrow{\quad b_2 \quad}$

Bob



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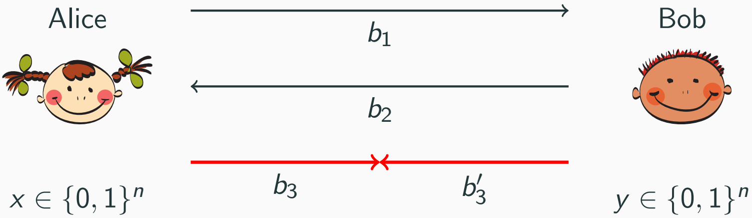

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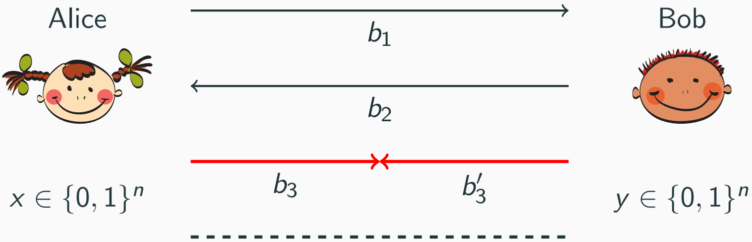

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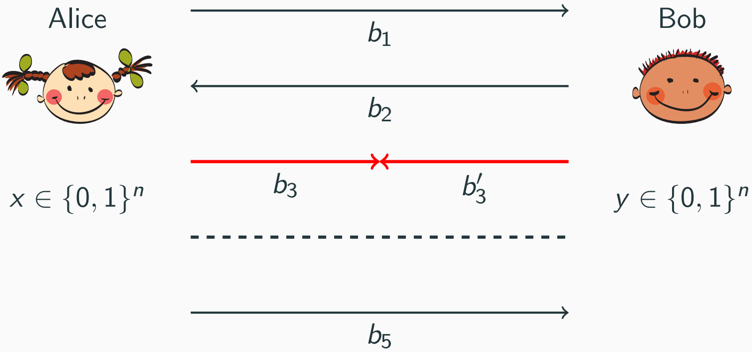

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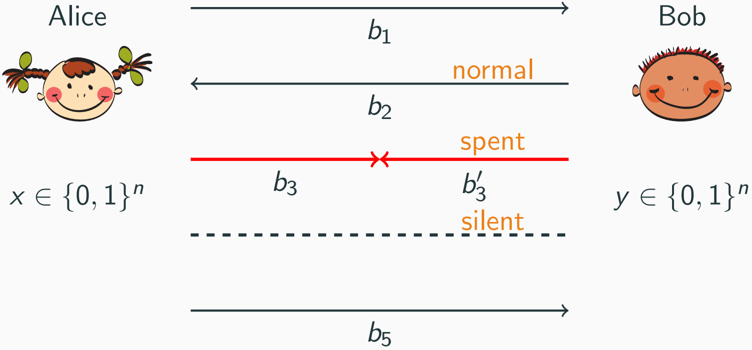

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Types of rounds

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2. Spent round: both players send.
3. Silent round: both players receive.

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3. Silent round: both players receive.

We consider three variants of how silent rounds work.

- **Half-duplex with silence**: the players receive some special symbol (i.e., silence), neither 0 nor 1.
- **Half-duplex with zero**: the players receive 0 (indistinguishable from normal round).
- **Half-duplex with an adversary**: the players receive bits chosen by an adversary (or some noise).

More formal

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$\{\texttt{send(0)}, \texttt{send(1)}, \texttt{receive}\}$.

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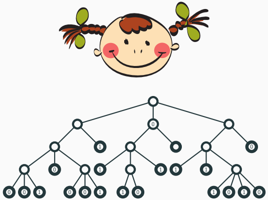
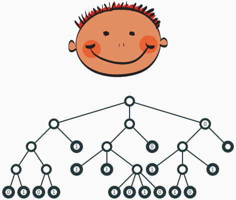
- In the model with zero the possible events are:
  $\{\texttt{send(1)}, \texttt{receive(0)}, \texttt{receive(1)}\}$
- In the model with an adversary the possible events are:
  $\{\texttt{send(0)}, \texttt{send(1)}, \texttt{receive(0)}, \texttt{receive(1)}\}$
  (in silent rounds the adversary chooses which bits the players receive)

Half-duplex communication protocol

Half-duplex communication protocol consists of two trees.





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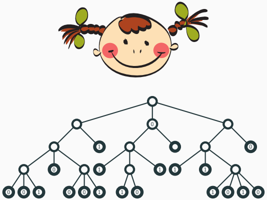

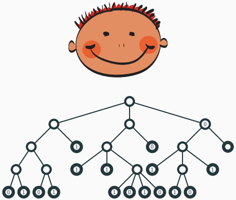

Half-duplex communication complexity of f with silence, with zero, with an adversary is a minimal depth of a protocol for f

- with silence, denoted $D_s^{hd}(f)$,
- with zero, denoted $D_0^{hd}(f)$,
- with an adversary, denoted $D_a^{hd}(f)$, respectively.

Simple observations

For every $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ the following holds.

1. $D^{hd}_s(f) \leq D^{hd}_0(f) \leq D^{hd}_a(f) \leq D(f)$ (follows from definitions).

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Note that multiplicative constants are important.

Functions

We study complexity of the following functions.

- Equality: $\mathrm{EQ}_n : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$,
  such that $\mathrm{EQ}_n(x, y) = 1 \iff x = y$.
- Inner product: $\mathrm{IP}_n : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$,
  such that $\mathrm{IP}_n(x, y) = \bigoplus_{i \in [n]} x_i y_i$.
- Disjointness: $\mathrm{DISJ}_n : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$,
  such that $\mathrm{DISJ}_n(x, y) = 1 \iff \forall i : x_i \neq 1 \vee y_i \neq 1$.

All this functions are hard in the classical model:

- $D(\mathrm{EQ}_n) \geq n$,
- $D(\mathrm{IP}_n) \geq n$,
- $D(\mathrm{DISJ}_n) \geq n$.

Methods for lower bounds

Methods

We consider three methods:

- Combinatorial rectangles.
- Round elimination.
- Internal information.

Combinatorial rectangles

- Every node v of a protocol corresponds to a combinatorial rectangle $R_v = X \times Y$, $X, Y \subseteq \{0,1\}^n$ of inputs: if $(x, y) \in R_v$ then the communication passes through the node v.
- A rectangle of some internal node v is equal to the union (disjoint) of rectangles in its children.
- Let μ be sub-additive measure: $\mu(\{0,1\}^n \times \{0,1\}^n) \geq \mu_r$, and for every leaf rectangle R_l, $\mu(R_l) \leq \mu_\ell$.
- Then the depth of any protocol is at least $\log_a(\mu_r/\mu_\ell)$, where a is the arity of the protocol trees.
- **Example of** μ: the minimal number of monochromatic rectangles that covers R.

Round elimination

Consider the first round of some protocol for f.

Alice\Bob	`send(0)`	`send(1)`	`receive`
`send(0)`	R_{00}	R_{01}	R_{0r}
`send(1)`	R_{10}	R_{11}	R_{1r}
`receive`	R_{r0}	R_{r1}	R_{rr}

Rectangle R is good for f if restricting f to R makes the first round useless. **Example**: $R = R_{00} \cup R_{01} \cup R_{0r}$.

Let μ be sub-additive measure: $\mu(\{0,1\}^n \times \{0,1\}^n) \geq \mu_r$ and for any leaf rectangle R_l, $\mu(R_l) \leq \mu_\ell$.

If for any rectangle R appearing in the protocol there is a good subrectangle for function $f \restriction R$ of measure at least $\alpha \cdot \mu(R)$ then the depth of the protocol is at least $\log_{1/\alpha}(\mu_r/\mu_\ell)$.

Internal information

Let $\mathcal{D}$ be a distribution over the domain of f.

$\mathcal{X}$ and $\mathcal{Y}$ — the marginal distributions over inputs.

Π_A and Π_B — the marginal distributions over transcripts.

An internal information cost of protocol $\mathcal{P}$ is $\mathrm{IC}_{\mathcal{D}}(\mathcal{P}) = I(\mathcal{X} : \Pi_B \mid \mathcal{Y}) + I(\mathcal{Y} : \Pi_A \mid \mathcal{X})$.

Lemma

Every leaf rectangle of a protocol for IP_n has size at most 2^n.

Lemma

If for any k-round half-duplex protocol $\mathcal{P}$ for IP_n with silence/zero/adversary, $\mathrm{IC}^k_{\mathcal{D}}(\mathcal{P}) \leq \alpha k$, then half-duplex complexity of IP_n with silence/zero/adversary is at least n/α.

Our results

Upper bounds

- For every $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$, $D_s^{hd}(f) \leq n/\log 3 + O(1) \leq n/1.58 + O(1)$.
- $D_s^{hd}(\mathrm{EQ}_n) \leq n/\log 5 + O(\log n)$.
- $D_s^{hd}(\mathrm{DISJ}_n) \leq n/2 + O(1)$.

Lower bounds

- $D_s^{hd}(\mathrm{EQ}_n) \geq \log_5 2^n = n/\log 5$ (combinatorial rectangles).
- $D_s^{hd}(\mathrm{IP}_n) \geq n/2$ (round elimination).
- $D_s^{hd}(\mathrm{IP}_n) \geq n/1.67$ (internal information).

Half-duplex with zero

Upper bounds

- $D_0^{hd}(\mathrm{EQ}_n) \leq n/\log 3 + O(\log n) \leq n/1.58 + O(\log n)$.

Lower bounds

- $D_0^{hd}(\mathrm{EQ}_n) \geq \log_3 2^n = n/\log 3$ (combinatorial rectangles).
- $D_0^{hd}(\mathrm{IP}_n) \geq n/\log \frac{2}{3-\sqrt{5}} > n/1.39$ (round elimination).
- $D_0^{hd}(\mathrm{IP}_n) \geq n/1.234$ (internal information).

Half-duplex with an adversary

Lower bounds

- $D_a^{hd}(\mathrm{EQ}_n) \geq \log_4 2^n = n/2$ (combinatorial rectangles).
- $D_a^{hd}(\mathrm{EQ}_n) \geq n/\log 2.5$ (round elimination).
- $D_a^{hd}(\mathrm{IP}_n) \geq n/\log \frac{7}{3}$ (round elimination).
- $D_a^{hd}(\mathrm{IP}_n) \geq n$ (internal information).

Half-duplex with an adversary

Lower bounds

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- $D_a^{hd}(\mathrm{IP}_n) \geq n/\log \frac{7}{3}$ (round elimination).
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Lower bound for KW game for parity

Let $R_{\oplus_n} = \{(x, y, i) \mid \oplus_n(x) = 0, \oplus_n(y) = 1, x_i \neq y_i\}$.

In the classical model $D(R_{\oplus_n}) \geq 2\log n$.

We proved that $D_a^{hd}(R_{\oplus_n}) \geq 2\log n$ using internal information.

Thanks for your attention!

Protocol for EQ_n with silence

Players encode inputs in an alphabet of size five $\{0, 1, 2, 3, 4\}$.

Symbol	**Alice**	**Bob**
0	`send(0)`	`receive`
1	`send(1)`	`receive`
2	`receive`	`send(0)`
3	`receive`	`send(1)`
4	`receive`	`receive`

Players can detect a mismatch in normal and silent rounds.

To check that there were no spent rounds, Alice sends the number of normal rounds she was receiving in.

Bob checks whether this number is equal to the number of rounds he was sending in.

Protocol for DISJ_n with silence

Alice and Bob process their inputs two bits per round.

Symbols	Alice	Bob
00	`send(0)`	`receive`
01	`receive`	`send(0)`
10	`receive`	`send(1)`
11	`receive`	`receive`

Bob tells Alice whether there was a silent round in which Bob's input was 11 (i.e., inputs are not disjoint).

Alice tells Bob whether she ever received 0 having 01 or 11, or received 1 having 10 or 11.

Lower bound for EQ_n with an adversary

Let $\mu(R) = \left|\{(x,x) \in R\}\right|$, the number of diagonal elements.

Consider the following set of five good rectangles:
$R_{spent} = R_{00} \cup R_{01} \cup R_{10} \cup R_{11}$, and four rectangles

$$R_{\bar{1}\bar{1}} = R_{00} \cup R_{0r} \cup R_{r0} \cup R_{rr}, \quad R_{\bar{0}\bar{1}} = R_{10} \cup R_{1r} \cup R_{r0} \cup R_{rr},$$
$$R_{\bar{1}\bar{0}} = R_{01} \cup R_{0r} \cup R_{r1} \cup R_{rr}, \quad R_{\bar{0}\bar{0}} = R_{11} \cup R_{1r} \cup R_{r1} \cup R_{rr},$$

Note that together all these good rectangles cover the entire rectangle R of possible input twice.

One of good rectangles has measure at least $2/5 \cdot \mu(R)$.

Hence $D_a^{hd}(\mathrm{EQ}_n) \geq n/\log 2.5$.

Summary and open questions

	EQ_n	IP_n	DISJ_n
D_s^{hd}	$\geq n/\log 5$ $\leq n/\log 5 + o(n)$	$\geq n/1.67$	$\leq n/2 + O(1)$
D_0^{hd}	$\geq n/\log 3$ $\leq n/\log 3 + o(n)$	$\geq n/1.234$	
D_a^{hd}	$\geq n/\log 2.5$	$\geq n$	

Open problems

1. Prove better upper and lower bounds.
2. Is there any $\alpha < 1$ such that for any function f, $D_0^{hd}(f) \leq \alpha n + o(n)$?
3. Is there any function f, such that at the same time $D(f) \geq n - o(n)$ and $D_a^{hd}(f) \leq \alpha n + o(n)$ for some $\alpha < 1$.

Motivation and related work

The Karchmer-Wigderson game for $f : \{0,1\}^n \to \{0,1\}$:
Alice is given $x \in f^{-1}(0)$, Bob is given $y \in f^{-1}(1)$, and they want to find an $i \in [n]$ such that $x_i \neq y_i$.

A multiplexer (aka *indexing function*) is a function $M_n : \{0,1\}^{2^n} \times \{0,1\}^n \to \{0,1\}$, such that $M_n(tt, x) = tt[x]$.

Consider Karchmer-Wigderson game for M_n:
Alice is given (tt_f, x), $x \in f^{-1}(0)$, Bob is given (tt_g, y), $y \in g^{-1}(1)$.

With a promise $f = g$ the players can use some protocol for f.

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What happens if the promise is broken?

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A multiplexer (aka *indexing function*) is a function $M_n : \{0,1\}^{2^n} \times \{0,1\}^n \to \{0,1\}$, such that $M_n(tt, x) = tt[x]$.

Consider Karchmer-Wigderson game for M_n:
Alice is given (tt_f, x), $x \in f^{-1}(0)$, Bob is given (tt_g, y), $y \in g^{-1}(1)$.

With a promise $f = g$ the players can use some protocol for f.

What happens if the promise is broken?

Similar models were studied by Impagliazzo and Williams (without "canceling" spent rounds), and by Efremenko, Kol, and Saxena (multi-party model with a noisy broadcast channel).