Around Hilbert's eighth and tenth problems

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http://logic.pdmi.ras.ru/~yumat

http://www-history.mcs.st-andrews.ac.uk/BigPictures/Hilbert_1900.jpeg

Mathematische Probleme

Vortrag, gehalten auf dem internationalen Mathematiker-Kongress, Paris, 1900

⋮

8. Primzahlenprobleme

⋮

10. Entscheidung der Lösbarkeit einer diophantischen Gleichung

⋮

8th Problem

8. Problems of prime numbers.

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10th Problem

10. Entscheidung der Lösbarkeit einer diophantischen Gleichung. Eine diophantische Gleichung mit irgendwelchen Unbekannten und mit ganzen rationalen Zahlkoefficienten sei vorgelegt: *man soll ein Verfahren angeben, nach welchen sich mittels einer endlichen Anzahl von Operationen entscheiden lässt, ob die Gleichung in ganzen rationalen Zahlen lösbar ist*.

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In the present talk, a Diophantine equation is an equation of the form

$$P(x_1, \ldots, x_m) = 0$$

where P is a polynomial with integer coefficients and the unknowns $x_1, \ldots, x_m$ can assume non-negative integer values only.

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Consider the set $\mathcal{M}$ such that

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Sets having such *representations* are called *Diophantine*

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Alfred Tarski question

Prove that the set of all prime numbers, or the set of all powers of 2, is not Diophantine

Julia Robinson predicates

Theorem (Julia Robinson [1952]) If there exists a two-parameter Diophantine equation

$$J(u, v, y_1, \ldots, y_n) = 0$$

such that

(∗) in every solution $u < v^v$;

(∗∗) for every k there exists a solution with $u > v^k$,

then exponetiation is Diophantine, that is, there exists a polynomial $A(a, b, c, w_1, \ldots, w_m)$ such that

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Relations between u and v satisfying (∗) and (∗∗) were named by Julia Robinson relations of exponential growth; later Martin Davis named them Julia Robinson predicates.

Listable Sets

Given a parametric Diophantine equation

$$P(a_1, \ldots, a_n, x_1, \ldots, x_m) = 0$$

we can effectively list all n-tuples from the Diophantine set $\mathcal{M}$ represented by this equation. Namely, we need only to look over, in some order, all $(n+m)$-tuples of possible values of all variables $a_1, \ldots, a_n$, $x_1, \ldots x_m$ and check every time whether the equality holds or not. As soon as it does, we put the tuple $\langle a_1, \ldots, a_n \rangle$ on the list of elements of $\mathcal{M}$. In this way every tuple from $\mathcal{M}$ will sooner or later appear on the list, maybe many times.

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Definition A set $\mathcal{M}$ of n-tuples of natural numbers is called listable (=effectively enumerable = semidecidable) if there is an algorithm which would print in some order, possibly with repetitions, all elements of the set $\mathcal{M}$.

Martin Davis's Conjecture

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Parametric Diophantine equation

↓ Number Theory

Description of the corresponding set

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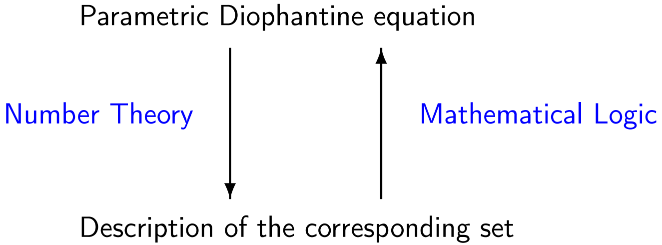

A Mile-Stone on the Way to Davis Conjecture

DPR-theorem (Martin Davis, Hilary Putnam, Julia Robinson [1961]). *Every listable listable set $\mathcal{M}$ has an exponential Diophantine representation of the form*

$$\langle a_1, \ldots, a_n \rangle \in \mathcal{M} \Longleftrightarrow \exists x_1 \ldots x_m$$
$$\{E_1(a_1, \ldots, a_n, x_1, \ldots, x_m) = E_2(a_1, \ldots, a_n, x_1, \ldots, x_m)\}$$

where $E_1(a_1, \ldots, a_n, x_1, \ldots, x_m)$ and $E_2(a_1, \ldots, a_n, x_1, \ldots, x_m)$ are expression constructed by combining the variables and particular natural numbers using the traditional rules of addition, multiplication and exponentiation.

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Missing link

After the work of Davis–Putnam–Robinson, in order to establish Davis's Conjecture in full generality it was sufficient to prove one of its very special cases, namely, to show that exponetiation is Diophantine, that is to find a particular Diophantine equation with 3 parameters such that

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And for this, thanks to 1952 work of Julia Robinson, it was sufficient to discover a Diophantine relation of exponential growth (Julia Robinson predicate).

Mathematical Reviews 1962, 24A, page 574, review A3061:

Davis, Martin; Putnam, Hilary; Robinson, Julia. The decision problem for exponential Diophantine equations. Ann. Math. (2), **74** 425–436 (1961).

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G.Kreisel

One Step More

J. Robinson. Unsolvable Diophantine problems. *Proceedings of the American Mathematical Society*, 22(2), 534–538, 1969.

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DPRM after Davis–Putnam–Robinson–Matiyasevich (sometimes DMPR theorem)

Computer verification of DPRM theorem

Karol Pąk
The Matiyasevich Theorem. Preliminaries
Formalized Mathematics, 25(4):315–322, 2017.
Diophantine sets. Preliminaries
Formalized Mathematics, 26(1):81–90, 2018.

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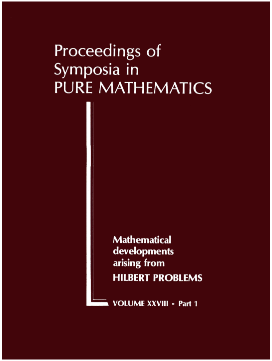
Hilbert Meets Isabelle: Formalisation of the DPRM Theorem in Isabelle
EasyChair Preprint no. 152, May 22, 2018

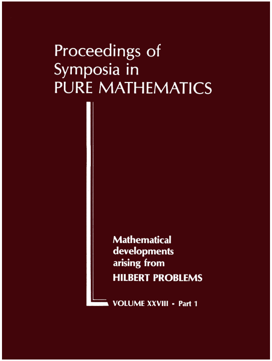
Dominique Larchey-Wendling and Yannick Forster
Hilbert's Tenth Problem in Coq
4th International Conference on Formal Structures for Computation and Deduction (FSCD 2019)
Leibniz International Proceedings in Informatics, No.27, 2019

AMS, DeKalb, Illinois, 1974



Mathematical developments arising from Hilbert problems

AMS, DeKalb, Illinois, 1974



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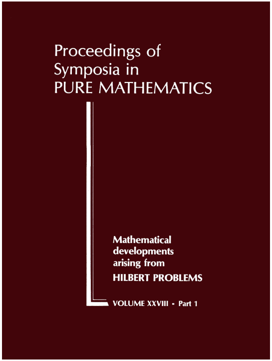

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Martin Davis, Yuri Matijasevič, and Julia Robinson

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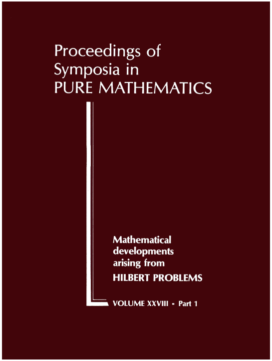

Mathematical developments arising from Hilbert problems

Proceedings of Symposia in Pure Mathematics, v. 28, 1976

MARTIN DAVIS, YURI MATIJASEVIČ, AND JULIA ROBINSON

Hilbert's tenth problem. Diophantine equations: positive aspects of a negative solution

Hilbert's 8th Problem

8. Problems of prime numbers.

"... to prove the correctness of an exceedingly important statement of Riemann, viz., *that the zero points of the function* $\zeta(s)$ *defined by the series*

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$$a \in \mathcal{M} \iff \exists x_1 \dots x_m \{G(a, x_1, \dots, x_m) = 0\}$$

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"... to attempt the rigorous solution of Goldbach's problem"

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So, a positive solution of Hilbert's tenth problem would allow us to know whether Goldbach's conjecture is true or not.

Hilbert's 8th Problem — Riemann's Hypothesis

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$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots$$

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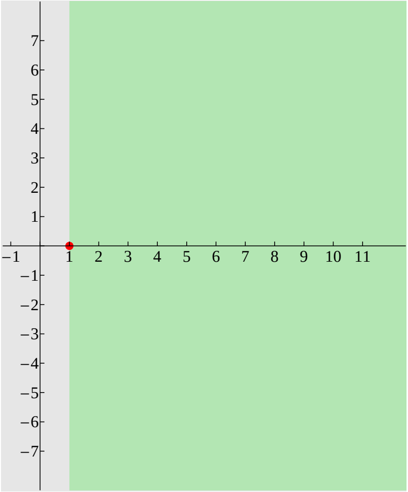
Riemann's zeta function

Dirichlet series:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$$s = \sigma + \mathrm{i}t$$

The series converges in the half-plane $\mathrm{Re}(s) > 1$ and defines a function that can be analytically extended to the entire complex plane except for the point $s = 1$, its only (and simple) pole.



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Euler identity $\equiv$ The Fundamental Theorem of Arithmetic

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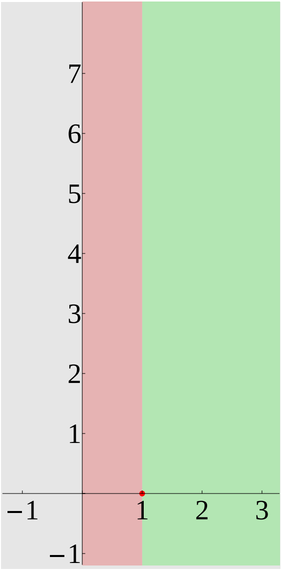

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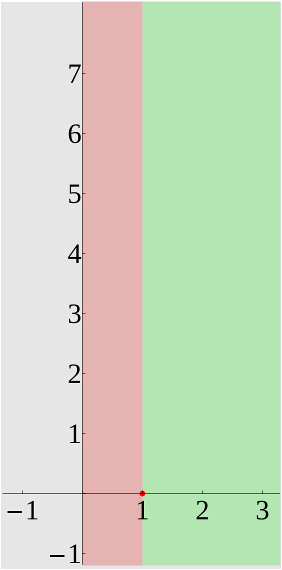

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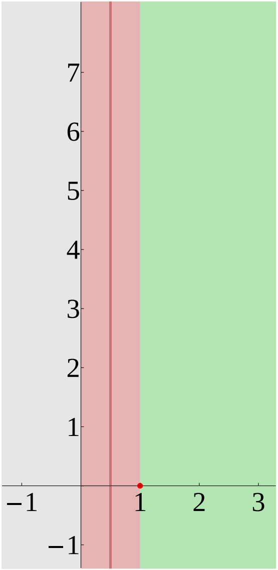

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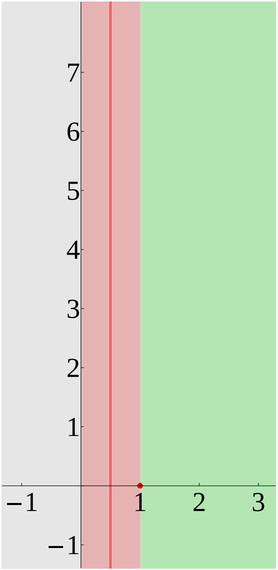
Equivalent formulation of RH.

$$\pi(x) - \mathrm{Li}(x) = O(x^{\frac{1}{2}} \log(x))$$



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K. Gödel: There exists an arithmetical formula equivalent to Riemann's Hypothesis

Arithmetical Hierarchy

$$\Pi_0^0 = \Sigma_0^0 = \{\phi(x_1, \ldots, x_k) | \text{ the validity of } \phi(x_1, \ldots, x_k) \text{ is algorithmically checkable}\}$$

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Where does RH lie in this hierarchy?

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Either RH $\Leftrightarrow 0 = 0$ or RH $\Leftrightarrow 0 = 1$

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Given what we know today, where in this hierarchy can we find a formula equivalent to RH?

Alan Turing thesis

A. M. Turing
Systems of logic based on ordinals
Proc. London Math. Soc., ser.2, vol. 45, 1939, pp. 161–228

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The Journal of Symbolic Logic, Vol. 23, No. 2 (Jun., 1958), pp. 155-182

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"... The applications of this observation depend, of course, on finding interesting propositions that can be formulated in the form above."

"... B(n) is primitive recursive by the construction above, and RH $\leftrightarrow (n)B(n)$."

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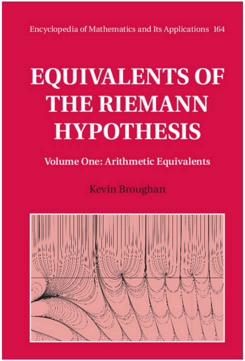
"...Turing had previously observed [31] that there is a primitive recursive $B(n, m)$ such that RH $\leftrightarrow (n)(Em)R(n, m)$ (in his argument he uses some special properties of the zeta function, while the argument above is quite general)."

Reformulations of Riemann's Hypothesis



Kevin Alfred Broughan

Equivalents of the Riemann Hypothesis

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Volume 2. Analytical Equivalents
Cambridge University Press, 2017

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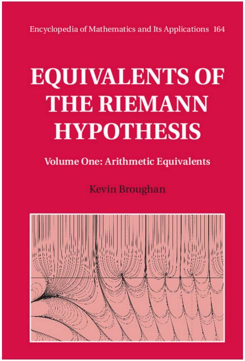

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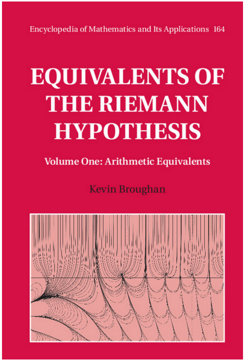

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Vol.1, p.241: "A subset $T \subset \mathbb{N}$ is computable if there is an algorithm to determine in a finite number of steps whether or not an arbitrary given natural number is a member of T [44]. From the theory of algorithms it follows that RH is decidable, i.e. its truth or negation are able to be proved."

Corollaries of DPRM theorem

$$\Pi_0^0 = \Sigma_0^0 = \{\phi(x_1, \ldots, x_m) | \text{ the validity of } \phi(x_1, \ldots, x_m) \text{ is algorithmically checkable}\}$$

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Proceedings of Symposia in Pure Mathematics
Volume 28, 1976

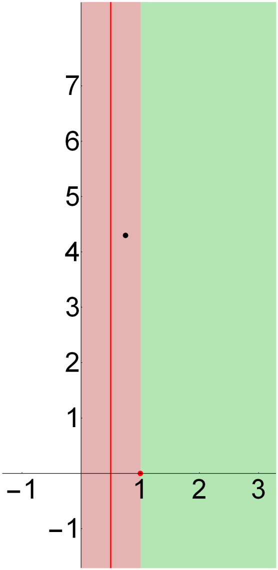
HILBERT'S TENTH PROBLEM. DIOPHANTINE EQUATIONS: POSITIVE ASPECTS OF A NEGATIVE SOLUTION

Martin Davis[1], Yuri Matijasevič, and Julia Robinson

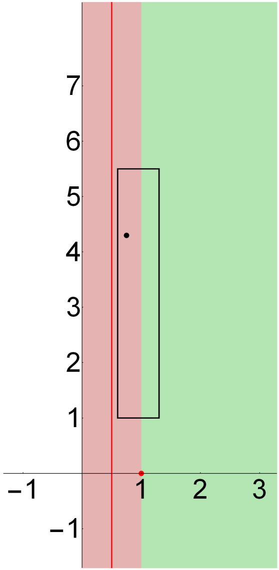
ABSTRACT

Applications (including the negative solution of Hilbert's tenth problem) and extensions are surveyed of the Main Theorem on Diophantine sets: Every listable (recursively enumerable) set is Diophantine. Key steps in the proof of the Main Theorem are outlined and applied to obtain prime representing polynomials, a universal Diophantine equation, and a sharp form of Gödel's incompleteness theorem. Many famous problems are reduced to the solvability of Diophantine equations. The number, size and effectiveness of solutions are discussed. Relationships are explored with the theory of algorithms (recursion theory), model theory, and algebraic number theory.

Cauchy integral



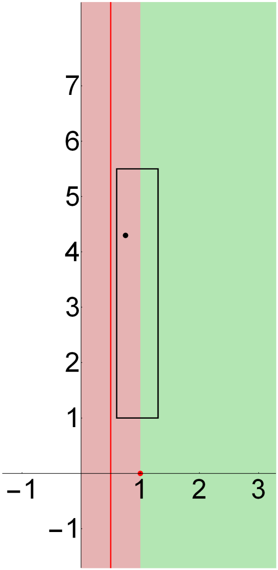
Cauchy integral



Cauchy integral

The number of zeros of $\zeta(s)$ inside the rectangular is equal to

$$\frac{1}{2\pi i} \oint \frac{\zeta'(s)}{\zeta(s)} \, ds$$



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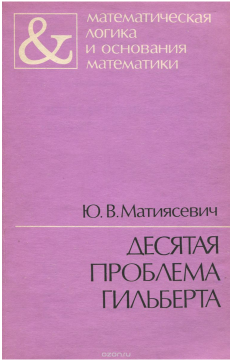
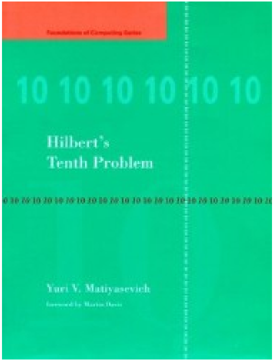
Theorem (H. N. Shapiro, [1974])

$$\mathrm{RH} \iff \forall m \left(\left| \psi_1(m) - \frac{m^2}{2} \right| < 6m\sqrt{m} \right)$$

Criterium of L. Schoenfeld





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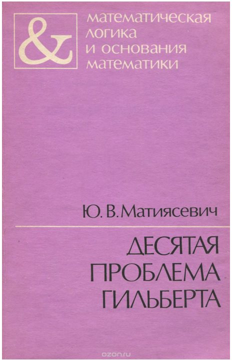

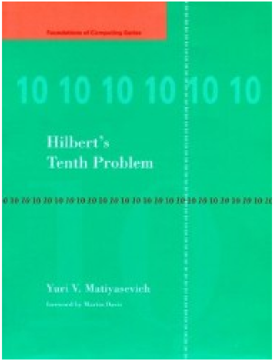

Theorem (L. Schoenfeld, [1976])

$$\mathrm{RH} \Leftrightarrow \forall n \left(n \geq 74 \Rightarrow |\psi(n) - n| < \frac{1}{8\pi}\sqrt{n}\ln(n)^2 \right)$$

More detailed presentations

Aran Nayebi
On the Riemann hypothesis and Hilbert's tenth problem
February 2012, Unpublished Manuscript,
http://web.stanford.edu/~anayebi/projects/RH_Diophantine.pdf.

J. M. Hernandez Caceres
The Riemann hypothesis and Diophantine equations, 2018.
Master's Thesis Mathematics, Mathematical Institute, University of Bonn

Yet another Π^0_1 formulation of Riemann's Hypothesis. I

J.-L. Nicolas
Petites valeurs de la fonction d'Euler
J. Number Theory, vol. 17, pp 375–388, 1983

Theorem.

$$\mathrm{RH} \Leftrightarrow \forall n \left(\mathrm{e}^{\gamma} \ln(\ln(N_n)) < \frac{N_n}{\phi(N_n)} \right),$$

where $\mathrm{e} = 2.71828\ldots$, N_n is the product of n first prime numbers, $\phi(m)$ is Euler's totient function (=the number of primes that are smaller than m and relatively prime to it), $\gamma = 0.577215\ldots$ is Euler constant:

$$\gamma = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \ln\left(1 + \frac{1}{k}\right)\right)$$

Yet another Π^0_1 formulation of Riemann's Hypothesis. II

G. Robin
Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann
J. Math. Pures Appl. (9) vol. 63, pp 187–213, 1984

Theorem.

$$\mathrm{RH} \Leftrightarrow \forall n \, (n \geq 5040 \Rightarrow \sigma(n) < \mathrm{e}^{\gamma} n \ln(\ln(n)))\,,$$

where $\sigma(n)$ is the sum of all divisors of n, $\gamma = 0.577215\ldots$ is Euler constant:

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Yet another Π_1^0 formulation of Riemann's Hypothesis. III

J. C. Lagarias
An elementary problem equivalent to the Riemann hypothesis
Am. Math. Mon. vol. 109, no. 6, pp 534–543, 2002

Theorem.

$$\mathrm{RH} \Leftrightarrow \forall n \left(\sigma(n) < H_n + \mathrm{e}^{H_n} \ln(H_n) \right),$$

where $\sigma(n)$ is the sum of all divisors of n, and $H_n = 1 + 1/2 + \cdots + 1/n$

A comparison of Π_1^0 formulations of Riemann's Hypothesis

RH

A comparison of Π_1^0 formulations of Riemann's Hypothesis

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A comparison of Π_1^0 formulations of Riemann's Hypothesis

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A comparison of Π^0_1 formulations of Riemann's Hypothesis

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&\iff \quad \forall n \left(\mathrm{e}^{\gamma} \ln(\ln(N_n)) < \frac{N_n}{\phi(N_n)} \right) \\
&\iff \quad \forall n \left(n \geq 5040 \Rightarrow \sigma(n) < \mathrm{e}^{\gamma} n \ln(\ln(n)) \right) \\
&\iff \quad \forall n \left(\sigma(n) < H_n + \mathrm{e}^{H_n} \ln(H_n) \right) \\
&\iff \quad \forall x_1 \ldots x_m R(x_1, \ldots, x_m) \neq 0
\end{aligned}
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Yet another Π^0_1 formulation of Riemann's Hypothesis. IV

Theorem (Matiyasevich [2018]). *Consider the following system of conditions:*

$$2^\ell \le n < 2^{\ell+1}, \qquad 2^m \le q < 2^{m+1},$$

$$s = \frac{B^{n+1}\left(B^{(n+1)n} - n - 1\right) + n}{\left(B^{n+1}-1\right)^2}, \qquad t = \frac{(2^m-1)\left(B^{n^2}-1\right)}{B^n - 1},$$

$$\binom{t}{r} \equiv 1 \pmod 2, \qquad rs - u \equiv \frac{B^{n^2-n}(B^n-1)}{B-1} q \pmod{B^{n^2}},$$

$$u = \mathrm{rem}\left(rs, B^{n^2-n}\right), \qquad p = \mathrm{rem}(r, B^n+1), \qquad mp < nq - 15\ell^2 q\sqrt{n},$$

where B denotes $2^{\ell+m+1}$.

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where B denotes $2^{\ell+m+1}$.
(A) If Riemann's Hypothesis is true, then the above system of conditions has no solution in positive integers $\ell, m, n, p, q, r, s, t, u$.

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where B denotes $2^{\ell+m+1}$.
(A) If Riemann's Hypothesis is true, then the above system of conditions has no solution in positive integers $\ell, m, n, p, q, r, s, t, u$.
(B) If Riemann's Hypothesis is not true, then the above system has infinitely many such solutions.

Yet another Π_1^0 formulation of Riemann's Hypothesis. V

A. A. Norkin

A Diophantine equation the unsolvability of which is equivalent to the Riemann Hypothesis

Bachelor thesis, Moscow, 2019

Yet another Π^0_1 formulation of Riemann's Hypothesis. V

A. A. Norkin

A Diophantine equation the unsolvability of which is equivalent to the Riemann Hypothesis

Bachelor thesis, Moscow, 2019

The equation has 193 unknowns