Around Hilbert's eighth and tenth problems

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http://logic.pdmi.ras.ru/~yumat

23 Hilbert's problems

http://www-history.mcs.st-andrews.ac.uk/BigPictures/Hilbert_1900.jpeg

Mathematische Probleme

Vortrag, gehalten auf dem internationalen Mathematiker-Kongress, Paris, 1900

⋮

8. Primzahlenprobleme

⋮

10. Entscheidung der Lösbarkeit einer diophantischen Gleichung

⋮

8th Problem

8. Problems of prime numbers.

"... to prove the correctness of an exceedingly important statement of Riemann, viz., *that the zero points of the function* $\zeta(s)$ *defined by the series*

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots$$

all have the real part $\frac{1}{2}$, except the well-known negative integral real zeros."

"... to attempt the rigorous solution of Goldbach's problem, viz., whether every even integer is expressible as the sum of two positive prime numbers"

"... to attack the well-known question, whether there are an infinite number of pairs of prime numbers with the difference 2"

10th Problem

10. Entscheidung der Lösbarkeit einer diophantischen Gleichung. Eine diophantische Gleichung mit irgendwelchen Unbekannten und mit ganzen rationalen Zahlkoefficienten sei vorgelegt: *man soll ein Verfahren angeben, nach welchen sich mittels einer endlichen Anzahl von Operationen entscheiden lässt, ob die Gleichung in ganzen rationalen Zahlen lösbar ist*.

10. Determination of the Solvability of a Diophantine Equation. Given a Diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: *To devise a process according to which it can be determined by a finite number of operations whether the equation is solvable in rational integers*.

What are Diophantine equations?

10. Determination of the Solvability of a Diophantine Equation. Given a Diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: *To devise a process according to which it can be determined by a finite number of operations whether the equation is solvable in rational integers*.

In the present talk, a Diophantine equation is an equation of the form

$$P(x_1, \ldots, x_m) = 0$$

where P is a polynomial with integer coefficients and the unknowns $x_1, \ldots, x_m$ can assume non-negative integer values only.

Diophantine Sets

10. Determination of the Solvability of a Diophantine Equation. Given a Diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: *To devise a process according to which it can be determined by a finite number of operations whether the equation is solvable in rational integers.*

$$P(a_1, \ldots, a_n, x_1, \ldots, x_m) = 0$$

P is a polynomial with integer coefficients, the variables of which are split into two groups:

the parameters $a_1, \ldots, a_n$ and the unknowns $x_1, \ldots, x_m$.

Consider the set $\mathcal{M}$ such that

$$\langle a_1, \ldots, a_n \rangle \in \mathcal{M} \Longleftrightarrow \exists x_1 \ldots x_m \{P(a_1, \ldots, a_n, x_1, \ldots, x_m) = 0\}$$

Sets having such *representations* are called *Diophantine*

Examples of Diophantine sets

The set of all squares:

$$a - x^2 = 0.$$

Is the set of all non-negative integer that are not full squares Diophantine? – EASY

The set of all composite numbers

$$a - (x_1 + 2)(x_2 + 2) = 0.$$

Is the set of all prime numbers Diophantine? – DIFFICULT

Te set of all positive integers which are not powers of 2

$$a - (2x_1 + 3)(x_2 + 1) = 0.$$

Is the set of all powers of 2 Diophantine? – DIFFICULT

Alfred Tarski question

Prove that the set of all prime numbers, or the set of all powers of 2, is not Diophantine

Julia Robinson predicates

Theorem (Julia Robinson [1952]) If there exists a two-parameter Diophantine equation

$$J(u, v, y_1, \ldots, y_n) = 0$$

such that

(∗) in every solution $u < v^v$;

(∗∗) for every k there exists a solution with $u > v^k$,

then exponetiation is Diophantine, that is, there exists a polynomial $A(a, b, c, w_1, \ldots, w_m)$ such that

$$a^b = c \Longleftrightarrow \exists z_1 \ldots z_m \{A(a, b, c, w_1, \ldots, w_m) = 0\}$$

Relations between u and v satisfying (∗) and (∗∗) were named by Julia Robinson relations of exponential growth; later Martin Davis named them Julia Robinson predicates.

Listable Sets

Given a parametric Diophantine equation

$$P(a_1, \dots, a_n, x_1, \dots, x_m) = 0$$

we can effectively list all n-tuples from the Diophantine set $\mathcal{M}$ represented by this equation. Namely, we need only to look over, in some order, all $(n+m)$-tuples of possible values of all variables $a_1, \dots, a_n$, $x_1, \dots x_m$ and check every time whether the equality holds or not. As soon as it does, we put the tuple $\langle a_1, \dots, a_n \rangle$ on the list of elements of $\mathcal{M}$. In this way every tuple from $\mathcal{M}$ will sooner or later appear on the list, maybe many times.

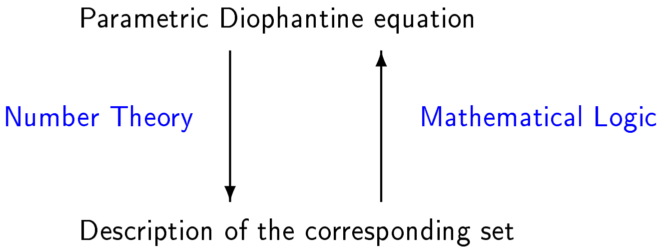
Definition A set $\mathcal{M}$ of n-tuples of natural numbers is called listable (=effectively enumerable = semidecidable) if there is an algorithm which would print in some order, possibly with repetitions, all elements of the set $\mathcal{M}$.

Martin Davis's Conjecture

Evident fact. *Every Diophantine set is listable.*

That is, if a set is not listable, then it cannot be Diophantine. Martin Davis conjectures that this is the only obstacle for a set to be Diophantine:

Martin Davis's Conjecture (1950's) *Every listable set is Diophantine.*



A Mile-Stone on the Way to Davis Conjecture

DPR-theorem (Martin Davis, Hilary Putnam, Julia Robinson [1961]). *Every listable listable set $\mathcal{M}$ has an exponential Diophantine representation of the form*

$$\langle a_1, \ldots, a_n \rangle \in \mathcal{M} \Longleftrightarrow \exists x_1 \ldots x_m$$
$$\{E_1(a_1, \ldots, a_n, x_1, \ldots, x_m) = E_2(a_1, \ldots, a_n, x_1, \ldots, x_m)\}$$

where $E_1(a_1, \ldots, a_n, x_1, \ldots, x_m)$ and $E_2(a_1, \ldots, a_n, x_1, \ldots, x_m)$ are expression constructed by combining the variables and particular natural numbers using the traditional rules of addition, multiplication and exponentiation.

Corollary. *The analogue of Hilbert's tenth problem for exponential Diophantine equations is undecidable.*

Missing link

After the work of Davis–Putnam–Robinson, in order to establish Davis's Conjecture in full generality it was sufficient to prove one of its very special cases, namely, to show that exponetiation is Diophantine, that is to find a particular Diophantine equation with 3 parameters such that

$$a^b = c \iff \exists z_1 \dots z_m \{A(a, b, c, w_1, \dots, w_m) = 0\}$$

And for this, thanks to 1952 work of Julia Robinson, it was sufficient to discover a Diophantine relation of exponential growth (Julia Robinson predicate).

Mathematical Reviews 1962, 24A, page 574, review A3061:

Davis, Martin; Putnam, Hilary; Robinson, Julia. The decision problem for exponential Diophantine equations. Ann. Math. (2), **74** 425–436 (1961).

> *... These results are superficially related to Hilbert's tenth problem on (ordinary, i.e., non-exponential) Diophantine equations. The proof of the authors'results, though very elegant, does not use recondite facts in the theory of numbers nor in the theory of r.e. [recursively enumerable] sets, and so it is likely that the present result is not closely connected with Hilbert's tenth problem...*

G.Kreisel

One Step More

J. Robinson. Unsolvable Diophantine problems. *Proceedings of the American Mathematical Society*, 22(2), 534–538, 1969.

Theorem. *If an infinite set of prime numbers is Diophantine, then Davis's Conjecture is true.*

Theorem. *If an infinite set of powers of* 2 *is Diophantine, then Davis's Conjecture is true.*

Реферативный журнал Математика

Russian counterpart to

- *Zentralblatt für Mathematik*
- *Mathematical Reviews*

Final step

The first example of Diophantine Julia Robinson predicate:

$$u = F_{2v}$$

$$F_0 = 0, \qquad F_1 = 1, \qquad F_{n+1} = F_n + F_{n-1}$$

DPRM Theorem [1970]. *Every listable set is Diophantine, that is, the notions of Diophantine set (from Number Theory) and of listable set (from Computability Theory) coincide.*

DPRM after Davis–Putnam–Robinson–Matiyasevich (sometimes DMPR theorem)

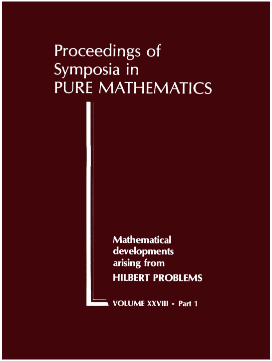
Computer verification of DPRM theorem

Karol Pąk
The Matiyasevich Theorem. Preliminaries
Formalized Mathematics, 25(4):315–322, 2017.
Diophantine sets. Preliminaries
Formalized Mathematics, 26(1):81–90, 2018.

Benedikt Stock, Abhik Pal, Maria Antonia Oprea, Yufei Liu, Malte Sophian Hassler, Simon Dubischar, Prabhat Devkota, Yiping Deng, Marco David, Bogdan Ciurezu, Jonas Bayer and Deepak Aryal
Hilbert Meets Isabelle: Formalisation of the DPRM Theorem in Isabelle
EasyChair Preprint no. 152, May 22, 2018

Dominique Larchey-Wendling and Yannick Forster
Hilbert's Tenth Problem in Coq
4th International Conference on Formal Structures for Computation and Deduction (FSCD 2019)
Leibniz International Proceedings in Informatics, No.27, 2019

AMS, DeKalb, Illinois, 1974



Mathematical developments arising from Hilbert problems

Proceedings of Symposia in Pure Mathematics, v. 28, 1976

Martin Davis, Yuri Matijasevic, and Julia Robinson

Hilbert's tenth problem. Diophantine equations: positive aspects of a negative solution

Hilbert's 8th Problem

8. Problems of prime numbers.

"... to prove the correctness of an exceedingly important statement of Riemann, viz., *that the zero points of the function* $\zeta(s)$ *defined by the series*

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots$$

all have the real part $\frac{1}{2}$, except the well-known negative integral real zeros."

"... to attempt the rigorous solution of Goldbach's problem, viz., whether every even integer is expressible as the sum of two positive prime numbers"

"... to attack the well-known question, whether there are an infinite number of pairs of prime numbers with the difference 2"

Hilbert's 8th Problem — Goldbach's Conjecture

8. Problems of prime numbers.
"... to attempt the rigorous solution of Goldbach's problem"

Conjecture (Ch. Goldbach [1742]). *Every even integer greater than 2 is the sum of two prime numbers.*

The set $\mathcal{M}$ of *counterexamples* to Goldbach's conjecture (i.e., even numbers greater than 2 not being the sum of two primes) is listable and hence we can construct its Diophantine representation

$$a \in \mathcal{M} \iff \exists x_1 \dots x_m \{G(a, x_1, \dots, x_m) = 0\}$$

Thus, Goldbach's conjecture is equivalent to the statement that the Diophantine equation

$$G(x_0, x_1, \dots, x_m) = 0$$

has no solution.

So, a positive solution of Hilbert's tenth problem would allow us to know whether Goldbach's conjecture is true or not.

Hilbert's 8th Problem — Riemann's Hypothesis

8. Problems of prime numbers. "... to prove the correctness of an exceedingly important statement of Riemann, viz., *that the zero points of the function* $\zeta(s)$ *defined by the series*

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots$$

all have the real part $\frac{1}{2}$, except the well-known negative integral real zeros."

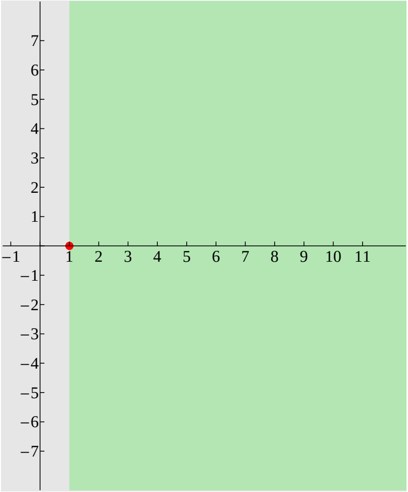
Riemann's zeta function

Dirichlet series:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$$s = \sigma + \mathrm{i}t$$

The series converges in the half-plane $\mathrm{Re}(s) > 1$ and defines a function that can be analytically extended to the entire complex plane except for the point $s = 1$, its only (and simple) pole.



Euler identity $\equiv$ The Fundamental Theorem of Arithmetic

Theorem (L. Euler [1737])

$$\begin{aligned}\zeta(s) &= 1^{-s} + 2^{-s} + \cdots + n^{-s} + \ldots \\ &= \prod_{p \text{ is prime}} \frac{1}{1-p^{-s}}\end{aligned}$$

Proof.

$$\begin{aligned}\prod_{p \text{ is prime}} \frac{1}{1-p^{-s}} &= \prod_{p \text{ is prime}} \left(1 + p^{-s} + p^{-2s} + p^{-3s} + \ldots\right) \\ &= 1^{-s} + 2^{-s} + 3^{-s} + \cdots + n^{-s} + \ldots\end{aligned}$$

Distribution of Prime Numbers

$$\pi(x) = \text{the number of primes not exceeding } x$$

C. F. Gauss conjectured that

$$\pi(x) \approx \mathrm{Li}(x) = \int_2^x \frac{1}{\ln(t)}\,\mathrm{d}t \approx \frac{x}{\ln(x)}$$

Theorem (B. Riemann [1859].)

$$\pi(x) = \mathrm{Li}(x) - \frac{1}{2}\mathrm{Li}(x^{\frac{1}{2}}) + \sum_{\zeta(\rho)=0} \mathrm{Li}(x^{\rho}) + \text{smaller terms}$$

Theorem (J. Hadamard, Ch. de la Vallee Poussin, [1896, independently])

$$\frac{\pi(x)}{x/\ln(x)} \to_{x\to\infty} 1 \qquad \frac{\pi(x)}{\mathrm{Li}(x)} \to_{x\to\infty} 1$$

Riemann's Hypothesis

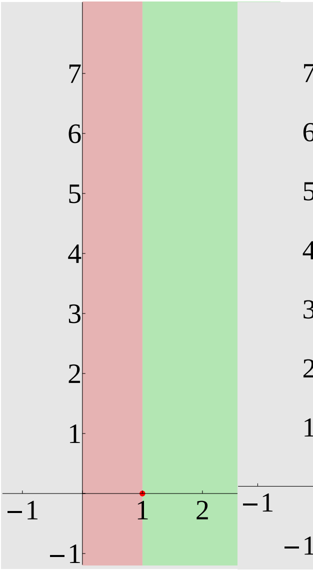
Euler: $0 = \zeta(-2) = \cdots = \zeta(-2m) = \ldots$

Theorem (Riemann [1859]). *All non-real zeros of* $\zeta(s)$ *lie in the critical strip* $0 \leq \mathrm{Re}(s) \leq 1$.

Riemann's Hypothesis (RH). *All non-real zeros of* $\zeta(s)$ *lie on the critical line* $\mathrm{Re}(s) = \frac{1}{2}$.

Equivalent formulation of RH.

$$\pi(x) - \mathrm{Li}(x) = O(x^{\frac{1}{2}} \log(x))$$



Gödel arithmetization

Equivalent formulation of RH:

$$\pi(x) - \mathrm{Li}(x) = O(x^{\frac{1}{2}} \log(x)) \qquad \mathrm{Li}(x) = \int_2^x \frac{1}{\ln(t)}\,\mathrm{d}t$$

K. Gödel: There exists an arithmetical formula equivalent to Riemann's Hypothesis

Arithmetical Hierarchy

$$\Pi^0_0 = \Sigma^0_0 = \{\phi(x_1, \ldots, x_k) | \text{ the validity of } \phi(x_1, \ldots, x_k) \text{ is algorithmically checkable}\}$$

$$\Pi^0_1 = \{\forall x_1 \ldots x_m \phi | \phi \in \Sigma^0_0\} \qquad \Sigma^0_1 = \{\exists x_1 \ldots x_m \phi | \phi \in \Pi^0_0\}$$

$$\Pi^0_2 = \{\forall x_1 \ldots x_m \phi | \phi \in \Sigma^0_1\} \qquad \Sigma^0_2 = \{\exists x_1 \ldots x_m \phi | \phi \in \Pi^0_1\}$$

$$\vdots \qquad \vdots$$

$$\Pi^0_{n+1} = \{\forall x_1 \ldots x_m \phi | \phi \in \Sigma^0_n\} \qquad \Sigma^0_{n+1} = \{\exists x_1 \ldots x_m \phi | \phi \in \Pi^0_n\}$$

$$\vdots \qquad \vdots$$

Where does RH lie in this hierarchy? RH$\in \Pi^0_0 = \Sigma^0_0$

Either RH $\Leftrightarrow 0 = 0$ or RH $\Leftrightarrow 0 = 1$

Given what we know today, where in this hierarchy can we find a formula equivalent to RH?

Alan Turing thesis

A. M. Turing
Systems of logic based on ordinals
Proc. London Math. Soc., ser.2, vol. 45, 1939, pp. 161–228

3. Number-theoretic theorems

By a number-theoretic theorem we shall mean a theorem of the form "$\theta(x)$ vanishes for infinitely many natural numbers x", where $\theta(x)$ is a primitive recursive function.

... Without going so far as this, I should assert that theorems of this kind have an importance which makes it worth while to give them special consideration.

Theorem. RH$\in \Pi^0_2 = \{\forall x_1 \dots x_m \exists y_1 \dots y_n \phi | \phi \in \Sigma^0_0\}$.

General technique of G. Kreisel

G. Kreisel

Mathematical Significance of Consistency Proofs

The Journal of Symbolic Logic, Vol. 23, No. 2 (Jun., 1958), pp. 155-182

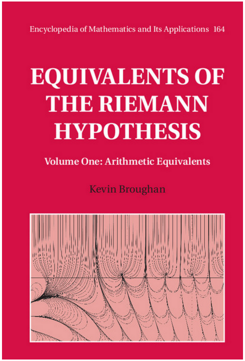
"... The applications of this observation depend, of course, on finding interesting propositions that can be formulated in the form above."

"... B(n) is primitive recursive by the construction above, and RH $\leftrightarrow (n)B(n)$."

Theorem. RH$\in \Pi^0_1 = \{\forall x_1 \ldots x_m \phi | \phi \in \Sigma^0_0\}$.

"...Turing had previously observed [31] that there is a primitive recursive $B(n, m)$ such that RH $\leftrightarrow (n)(Em)R(n, m)$ (in his argument he uses some special properties of the zeta function, while the argument above is quite general)."

Reformulations of Riemann's Hypothesis



Kevin Alfred Broughan

Equivalents of the Riemann Hypothesis

Volume 1. Arithmetic Equivalents
Volume 2. Analytical Equivalents
Cambridge University Press, 2017

Vol.1, p.241: "A subset $T \subset \mathbb{N}$ is computable if there is an algorithm to determine in a finite number of steps whether or not an arbitrary given natural number is a member of T [44]. From the theory of algorithms it follows that RH is decidable, i.e. its truth or negation are able to be proved."

Corollaries of DPRM theorem

$$\Pi_0^0 = \Sigma_0^0 = \{\phi(x_1,\ldots,x_m) | \text{ the validity of } \phi(x_1,\ldots,x_m) \text{ is algorithmically checkable}\}$$

$$\Pi_1^0 = \{\forall x_1 \ldots x_m \phi | \phi \in \Sigma_0^0\}$$

Corollaries of DPRM theorem. *For every formula* $\phi(a_1,\ldots,a_k)$ *from* Π_1^0 *we can effectively construct a polynomial* $P(a_1,\ldots,a_n,x_1,\ldots,x_k)$ *with integer coefficients such that*

$$\phi(a_1,\ldots,a_m) \iff \forall x_1 \ldots x_m P(a_1,\ldots,a_n,x_1,\ldots,x_m) \neq 0;$$

in particular, we can construct a specific polynomial $R(x_1,\ldots,x_m)$ *with integer coefficients such that*

$$\begin{aligned} RH &\iff \forall x_1 \ldots x_m R(x_1,\ldots,x_m) \neq 0 \\ &\iff \neg\exists x_1 \ldots x_m R(x_1,\ldots,x_m) = 0 \end{aligned}$$

Proceedings of Symposia in Pure Mathematics
Volume 28, 1976

HILBERT'S TENTH PROBLEM. DIOPHANTINE EQUATIONS: POSITIVE ASPECTS OF A NEGATIVE SOLUTION

Martin Davis[1], Yuri Matijasevič, and Julia Robinson

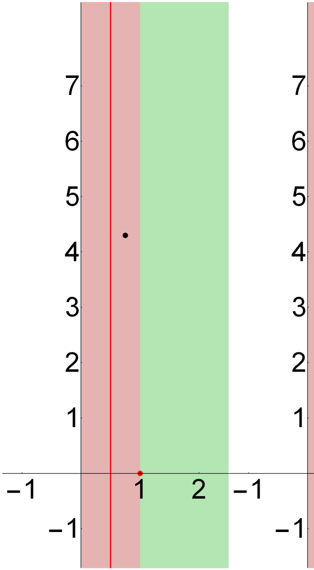
ABSTRACT

Applications (including the negative solution of Hilbert's tenth problem) and extensions are surveyed of the Main Theorem on Diophantine sets: Every listable (recursively enumerable) set is Diophantine. Key steps in the proof of the Main Theorem are outlined and applied to obtain prime representing polynomials, a universal Diophantine equation, and a sharp form of Gödel's incompleteness theorem. Many famous problems are reduced to the solvability of Diophantine equations. The number, size and effectiveness of solutions are discussed. Relationships are explored with the theory of algorithms (recursion theory), model theory, and algebraic number theory.

Cauchy integral

The number of zeros of $\zeta(s)$ inside the rectangular is equal to

$$\frac{1}{2\pi i}\oint \frac{\zeta'(s)}{\zeta(s)}\,\mathrm{d}s$$



Chebyshev function $\psi(x)$

$$\pi(x) = \sum_{\substack{p \le x \\ p \text{ is a prime } p}} 1 \qquad \psi(x) = \sum_{\substack{q \le x \\ q \text{ is a power} \\ \text{of a prime } p}} \ln(p)$$

$$= \ln(\mathrm{LCM}(1, 2, \ldots, \lfloor x \rfloor))$$

$$\pi(x) \approx \frac{x}{\ln(x)} \qquad \psi(x) \approx x$$

$$\pi(x) = \mathrm{Li}(x) - \frac{1}{2}\mathrm{Li}(x^{\frac{1}{2}}) + \sum_{\zeta(\rho)=0} \mathrm{Li}(x^{\rho}) + \text{smaller terms}$$

$$\psi(x) = x - \sum_{\zeta(\rho)=0} \frac{x^{\rho}}{\rho} - \ln(2\pi)$$

$$\pi(x) - \mathrm{Li}(x) = O(x^{\frac{1}{2}} \log(x)) \qquad \psi(x) = x + O(\sqrt{x} \ln(x)^2)$$

Riemann's Hypothesis in terms of Chebyshev function $\psi(x)$

$$\psi(x) \quad = \quad \ln(\mathrm{LCM}(1, 2, ..., \lfloor x \rfloor))$$

$$\begin{aligned} \mathrm{RH} \quad & \Longleftrightarrow \quad \psi(n) = n + O(\sqrt{n}\ln(n)^2) \\ & \Longleftrightarrow \quad \exists c\, \forall n \left(|\psi(n) - n| \leq c\sqrt{n}\ln(n)^2 \right) \end{aligned}$$

$$\exists c \left(\mathrm{RH} \iff \forall n \big(|\psi(n) - n| \leq c\sqrt{n}\ln(n)^2 \big)\right)$$

Criterium of H. N. Shapiro

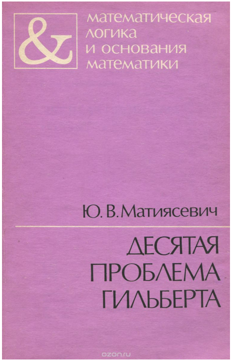
$$\psi(x) \quad = \quad \ln(\mathrm{LCM}(1, 2, ..., \lfloor x \rfloor))$$

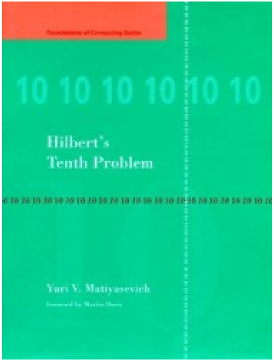
$$\psi_1(m) = \sum_{1 \le n < m} \psi(n) \qquad \psi(n) \approx n \Rightarrow \psi_1(m) \approx \frac{m^2}{2}$$

Theorem (H. N. Shapiro, [1974])

$$\mathrm{RH} \iff \forall m \left(\left| \psi_1(m) - \frac{m^2}{2} \right| < 6m\sqrt{m} \right)$$

Criterium of L. Schoenfeld





Theorem (L. Schoenfeld, [1976])

$$\mathrm{RH} \Leftrightarrow \forall n \left(n \geq 74 \Rightarrow |\psi(n) - n| < \frac{1}{8\pi}\sqrt{n}\ln(n)^2 \right)$$

More detailed presentations

Aran Nayebi
On the Riemann hypothesis and Hilbert's tenth problem
February 2012, Unpublished Manuscript,
`http://web.stanford.edu/~anayebi/projects/RH_Diophantine.pdf`.

J. M. Hernandez Caceres
The Riemann hypothesis and Diophantine equations, 2018.
Master's Thesis Mathematics, Mathematical Institute, University of Bonn

Yet another Π_1^0 formulation of Riemann's Hypothesis. I

J.-L. Nicolas
Petites valeurs de la fonction d'Euler
J. Number Theory, vol. 17, pp 375–388, 1983

Theorem.

$$\mathrm{RH} \Leftrightarrow \forall n \left(\mathrm{e}^{\gamma} \ln(\ln(N_n)) < \frac{N_n}{\phi(N_n)} \right),$$

where $\mathrm{e} = 2.71828\ldots$, N_n is the product of n first prime numbers, $\phi(m)$ is Euler's totient function (=the number of primes that are smaller than m and relatively prime to it), $\gamma = 0.577215\ldots$ is Euler constant:

$$\gamma = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \ln\left(1 + \frac{1}{k}\right)\right)$$

Yet another Π_1^0 formulation of Riemann's Hypothesis. II

G. Robin
Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann
J. Math. Pures Appl. (9) vol. 63, pp 187–213, 1984

Theorem.

$$\mathrm{RH} \Leftrightarrow \forall n \, (n \geq 5040 \Rightarrow \sigma(n) < \mathrm{e}^{\gamma} n \ln(\ln(n))) \, ,$$

where $\sigma(n)$ is the sum of all divisors of n, $\gamma = 0.577215\ldots$ is Euler constant:

$$\gamma = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \ln \left(1 + \frac{1}{k} \right) \right)$$

Yet another Π^0_1 formulation of Riemann's Hypothesis. III

J. C. Lagarias
An elementary problem equivalent to the Riemann hypothesis
Am. Math. Mon. vol. 109, no. 6, pp 534–543, 2002

Theorem.

$$\mathrm{RH} \Leftrightarrow \forall n \left(\sigma(n) < H_n + \mathrm{e}^{H_n} \ln(H_n)\right),$$

where $\sigma(n)$ is the sum of all divisors of n, and $H_n = 1 + 1/2 + \cdots + 1/n$

A comparison of Π_1^0 formulations of Riemann's Hypothesis

$$
\begin{aligned}
\text{RH} \quad & \iff \quad \forall m \left(\left| \psi_1(m) - \frac{m^2}{2} \right| < 6m\sqrt{m} \right) \\
& \iff \quad \forall n \left(n \geq 74 \Rightarrow |\psi(n) - n| < \frac{1}{8\pi}\sqrt{n}\ln(n)^2 \right) \\
& \iff \quad \forall n \left(\mathrm{e}^{\gamma} \ln(\ln(N_n)) < \frac{N_n}{\phi(N_n)} \right) \\
& \iff \quad \forall n \left(n \geq 5040 \Rightarrow \sigma(n) < \mathrm{e}^{\gamma} n \ln(\ln(n)) \right) \\
& \iff \quad \forall n \left(\sigma(n) < H_n + \mathrm{e}^{H_n} \ln(H_n) \right) \\
& \iff \quad \forall x_1 \ldots x_m R(x_1, \ldots, x_m) \neq 0
\end{aligned}
$$

Yet another Π_1^0 formulation of Riemann's Hypothesis. IV

Theorem (Matiyasevich [2018]). *Consider the following system of conditions:*

$$2^\ell \le n < 2^{\ell+1}, \qquad 2^m \le q < 2^{m+1},$$

$$s = \frac{B^{n+1}\left(B^{(n+1)n} - n - 1\right) + n}{\left(B^{n+1} - 1\right)^2}, \qquad t = \frac{(2^m - 1)\left(B^{n^2} - 1\right)}{B^n - 1},$$

$$\binom{t}{r} \equiv 1 \pmod 2, \qquad rs - u \equiv \frac{B^{n^2-n}(B^n - 1)}{B - 1} q \pmod{B^{n^2}},$$

$$u = \mathrm{rem}\left(rs, B^{n^2-n}\right), \qquad p = \mathrm{rem}(r, B^n + 1), \qquad mp < nq - 15\ell^2 q\sqrt{n},$$

where B denotes $2^{\ell+m+1}$.
(A) If Riemann's Hypothesis is true, then the above system of conditions has no solution in positive integers $\ell, m, n, p, q, r, s, t, u$.
(B) If Riemann's Hypothesis is not true, then the above system has infinitely many such solutions.

Yet another Π^0_1 formulation of Riemann's Hypothesis. V

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Bachelor thesis, Moscow, 2019

The equation has 193 unknowns