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UDK 519.1

ON A CERTAIN REPRESENTATION OF THE CHROMATIC POLYNOMIAL

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1. NOTATION. We introduce notation for certain standard notions in graph theory. (Definitions of some of these notions are given below.)

By a graph we mean a non-oriented graph, possibly with multiple edges and loops. $n(H)$ and $m(H)$ will denote the number of vertices and edges, respectively, of a graph H. Let $H \leq G$ mean that H is an edge subgraph of a graph G, and let $H \preceq G$ mean that graph H can be obtained from graph G by contracting some of its edges.

By $C(H, k)$ we denote the chromatic polynomial of graph H, that is, the number of proper colorings of the vertices of graph H in at most k colors, and by $F(H, k)$ we denote the flow polynomial of graph H, that is the number of flows modulo k having neither sources nor sinks and equal to 0 on none of the edges.

2. RESULTS. We shall prove one theorem and deduce three of its corollaries.

THEOREM. *For every graph G*

$$C(G, k) = \frac{(k-1)^{m(G)}}{k^{m(G)-n(G)}} \sum_{H \leq G} \frac{F(H, k)}{(1-k)^{m(H)}}. \tag{1}$$

EXAMPLE 1. Let G be a tree, then $m(G) = n(G) - 1$. If $H \leq G$ then $F(H,k) = 0$ with exception of the degenerate case $m(H) = n(H) = 0$ when $F(H,k) = 1$. Hence,

$$C(G,k) = k(k-1)^{m(G)}.$$

EXAMPLE 2. Let G be a simple circle, then $m(G) = n(G)$. If $H \leq G$ then $F(H,k)$ is different from zero only in two extreme cases: when $m(H) = m(G)$, that is when $H = G$, and when $m(H) = 0$. In the former case $F(H,k) = k-1$, in the latter case $F(H,k) = 1$; hence

$$C(G,k) = (-1)^{m(G)}(k-1) + (k-1)^{m(G)}.$$

In [1] the following representation was obtained:

$$C(G,k) = \frac{(k-1)^{m(G)}}{k^{m(G)-n(G)}} \sum_{H \leq G} \frac{w(H,k)}{(1-k)^{m(H)}}. \tag{2}$$

The function w was first defined as the sum of the values of a certain weight function with summation over all (that is, non necessary proper) colorings of graph G (formula 2.5 in [1]); later w was defined via a recurrent relation (Theorem VII in [1]). An easy analysis of the proofs shows that representations (1) and (2) termwise coincide, that is, always we have

$$F(H,k) = k^{m(H)-n(H)} w(H,k). \tag{3}$$

Thus the proposed theorem can be viewed as a relationship between the special function introduced in [1] and more traditional notions in graph theory.

The definition of the function w implied by (3) makes evident a number of its properties established in [1] to facilitate computation of the function:

- $w(H,k) = 0$ as long as G contains an isthmus (Theorems I and IV in [1]);
- $w(H,k) = w(H_1,k)w(H_2,k)$ provided that H consists of two parts without common vertices (Theorem II in [1]);
- $w(H,k) = w(H_1,k)w(H_2,k)/k$ provided that H_1 and H_2 have a single common vertex (Theorem III in [1]);

- $w(H', k) = w(H'', k)$ provided that H' and H'' are homeomorphic (Theorem V in [1]).

The transition from (2) to (1) is most interesting when G is a planar graph. In this case each subgraph H in (1) is also planar and we can find its geometric dual graph H^*. It is easy to check that

$$F(H, k) = C(H^*, k)/k \tag{4}$$

and hence we have

COROLLARY 1. For every planar graph G

$$C(G, k) = \frac{(k-1)^{m(G)}}{k^{m(G)-n(G)+1}} \sum_{H \leq G} \frac{C(H^*, k)}{(1-k)^{m(H)}}. \tag{5}$$

This result can be restated in a dual form:

COROLLARY 1. For every planar graph G

$$C(G, k) = \frac{(k-1)^{m(G^*)}}{k^{n(G^*)-s}} \sum_{L \leq G^*} \frac{C(L, k)}{(1-k)^{m(L)}}. \tag{6}$$

(This result shows, in particular, how one can find the chromatic polynomial of a connected planar graph G from its combinatory dual graph G^*, although the graph G itself isn't, in general, determined uniquely by G^*.)

If $m(H) \geq 1$ then it is easy to see that $C(H, k) \equiv 0 \pmod{k-1}$. This implies that, in the case when $m(G) > 1$, passing from (5) to congruence modulo $(k-1)^2$, we cam omit all summands except the one corresponding to the case $H = G$. Thus we have

COROLLARY 2. *If* $m > 1$ *then*

$$C(G, k) \equiv (-1)^m C(G^*, k) \pmod{(k-1)^2} \tag{7}$$

Putting here $k = 3$, we get

COROLLARY 3. If a planar graph G is different from the full graph K_2 and has exactly one (up to renaming of colors) proper coloring of vertices in three colors, then the graph G^* dual to graph G is also vertex colorable in three colors.

3.DEFINITIONS. Let G, H be graphs, and let $V(G)$, $V(H)$, $E(G)$, $E(H)$ be the corresponding sets of vertices and edges. We say that graph H is an edge subgraph of graph G (and write $H \leq G$) if $E(H) \subseteq E(G)$ and $V(H)$ consists of those and only those vertices of G that are incident to edges from $E(H)$. For the sake of validity of formula (1) we admit the case when $E(H)$ (and hence $V(H)$) is the empty set.

The operation of contracting graph G by edges connecting two adjacent vertices v' and v'' consists in removing those (and only those) edges and identifying vertices v' and v'' into a single vertex v; thus, if vertices v' and v'' had been connected to a vertex w by paths of l' and l'' edges respectively, then the new vertex v is connected to w by a path of $l' + l''$ edges. We say that graph H is a contraction of graph G (and write $H \preceq G$) if H can be obtained from G by a number, possibly zero, of edge contractions.

We take the ring R_k of residues modulo k as the standard set of k colors. By a vertex coloring of graph H we mean any function defined on $V(H)$ with values from R_k; a coloring is called proper if the ends of each edge have different colors. By $S(H,k)$ we denote the set of all colorings in k colors, and by $S^+(H,k)$ we denote the set of all proper colorings in k colors. In this notation $C(H,k) = |S^+(H,k)|$ is the cardinality of $S^+(H,k)$. It is well-known (see, for example, [2]) that for a fixed H, the function $C(H,k)$ is a polynomial of degree $n(H)$ with integer coefficients.

In order to be able to introduce flows on a graph H we need to fix some orientation of all edges, which will be done by denoting by e' and e'' the beginning and the end of an edge e respectively. When such an orientation is fixed, a flow modulo k on graph H is defined as any function defined on $E(H)$ with values from R_k. (It is supposed that if we change the orientation of some edges, we'll have to change the sign of the flow on those edges; all notions introduced below are invariant with respect to such transformations.) We say that a flow t is balanced if it has neither sources nor sinks, that is if for every vertex v

$$\sum_{e'=v} t(e) = \sum_{e''=v} t(e) \tag{8}$$

(the summation is performed in the ring R_k, that is, modulo k). The degree of degeneracy $d(t)$ of a flow t is defined as the number of edges e such that $t(e) = 0$; a flow t is called non-degenerate if $d(t) = 0$. By $T(H, k)$ we denote the set of all flows modulo k on a graph H, and by $T^{=}(H, k)$ we denote the set of all balanced flows. The flow polynomial $F(H, k)$ equal to the number of non-degenerate balanced flows was introduced in [3] (in [3] it was denoted $\phi(H, k)$; see also [4, Section 14C]).

If H is a plane graph, that is, we have fixed a mapping of it to a plane, then its geometric dual graph H^* is defined in the following way. Vertices of H^* correspond to the areas on which graph H cuts the plane, and the edges of H^* correspond to the edges of graph H: an edge e^* from $E(H^*)$ connects vertices v_1^* and v_2^* from $V(H^*)$ if the edge e dual to e^* separates the areas corresponding to vertices e_1^* and e_2^* (isthmuses of graph H correspond to loops in graph H^*).

4.PROOFS. Let G be an arbitrary graph, k be a positive integer. We will use the shorthand $V = V(G)$, $n = n(G)$, $S = S(G, k)$, and so on.

Let $\varepsilon = \cos(2\pi/k) + \mathrm{i}\sin(2\pi/k)$ be a primitive root of unity of degree k. Then for $r \in R_k$

$$\sum_{t=0}^{k-1} \varepsilon^{rt} = \begin{cases} k, & \text{if } r = 0, \\ 0, & \text{if } r \neq 0. \end{cases}$$

From this we get that for $s \in S$

$$\prod_{e \in E} \left(k - \sum_{t=0}^{k-1} \varepsilon^{(s(e')-s(e''))t} \right) = \begin{cases} k^m, & \text{if } s \in S^+, \\ 0, & \text{if } s \notin S^+. \end{cases} \tag{9}$$

Let

$$\delta(t) = \begin{cases} k-1, & \text{if } t = 0, \\ -1, & \text{if } t \neq 0. \end{cases}$$

then by (9)

$$k^m C(G, k) = \sum_{s \in S^+} k^m = \sum_{s \in S} \prod_{e \in E} \sum_{t=0}^{k-1} \delta(t) \varepsilon^{(s(e')-s(e''))t}.$$

Further we have:

$$\prod_{e \in E} \sum_{t=0}^{k-1} \delta(t) \varepsilon^{(s(e')-s(e''))t} = \sum_{t \in T} \prod_{e \in E} \delta(t(e)) \varepsilon^{(s(e')-s(e''))t(e)},$$

$$\prod_{e\in E}\delta(t(e))\varepsilon^{(s(e')-s(e''))t(e)} = \prod_{e\in E}\delta(t(e)) \times \prod_{e\in E}\varepsilon^{s(e')t(e)} \times \prod_{e\in E}\varepsilon^{-s(e'')t(e)},$$

$$\prod_{e\in E}\delta(t(e)) = (-1)^m(1-k)^d(t),$$

$$\begin{aligned}
\prod_{e\in E}\varepsilon^{s(e')t(e)} &= \prod_{\substack{e\in E\\ v\in V\\ e'=v}}\varepsilon^{s(v)t(e)}\\
&= \prod_{v\in V}\prod_{e'=v}\varepsilon^{s(v)t(e)}\\
&= \prod_{v\in V}\varepsilon^{\sum_{e'=v}s(v)t(e)}\\
&= \prod_{v\in V}\varepsilon^{\sum_{e'=v}t(e)\times s(v)}.
\end{aligned}$$

Similarly,

$$\prod_{e\in E}\varepsilon^{-s(e'')t(e)} = \prod_{v\in V}\varepsilon^{-\sum_{e''=v}t(e)\times s(v)},$$

so that,

$$\begin{aligned}
k^mC(G,k) &= \sum_{s\in S,\ t\in T}(-1)^m(1-k)^{d(t)}\prod_{v\in V}\varepsilon^{\left(\sum_{e'=v}t(e)-\sum_{e''=v}t(e)\right)s(v)}\\
&= (-1)^m\sum_{t\in T}(1-k)^{d(t)}\prod_{v\in V}\sum_{s=0}^{k-1}\varepsilon^{\left(\sum_{e'=v}t(e)-\sum_{e''=v}t(e)\right)s(v)}\\
&= (-1)^m\sum_{t\in T^=}(1-k)^{d(t)}k^n\\
&= (-1)^mk^n\sum_{t\in T^=}(1-k)^{d(t)}. \qquad (10)
\end{aligned}$$

For every flow t from $T^=$ we define the subgraph G_t as the graph obtained from G by removing those and only those edges e for which $t(e) = 0$; clearly, $d(t) = n(G) - m(G_t)$. It is easy to see that the restriction of a flow t on the graph G_t is a balanced non-degenerate flow on G_t, and, vice versa, for every balanced non-degenerate flow t_H on any spanning subgraph H there exists a unique flow t such that $G_t = H$ and t_H is the restriction of t on H. Thus

$$F(H,k) = \sum_{t \in T^{=}(G,k),\ G_t = H} 1.$$

Continuing from (10):

$$\begin{aligned}
k^m C(G,k) &= (-1)^m k^n \sum_{t \in T^{=}} (1-k)^{d(t)} \\
&= (-1)^m k^n \sum_{H \leq G} \sum_{t \in T^{=}(G,k),\ G_t = H} (1-k)^{d(t)} \\
&= (-1)^m k^n \sum_{H \leq G} (1-k)^{-m(H)} \sum_{t \in T^{=}(G,k),\ G_t = H} 1 \\
&= (-1)^m k^n \sum_{H \leq G} \frac{F(H,k)}{(1-k)^{m(H)}}.
\end{aligned}$$

The Theorem is proved.

Relation (4) is given in [3] without proof (see also [4, Section 14C]). For completeness we prove it now.

The addition operation of the ring R_k induces an addition in $T(H,k)$; namely, let $t_1 + t_2$ be such a flow that $(t_1+t_2)(e) = t_1(e) + t_2(e)$ for $t_1, t_2 \in T(H,k)$ and $e \in E(H)$. Similarly, the multiplication operation of R_k allows us to multiply the elements of $T(H,k)$ by the elements of this ring: $(rt)(e) = r \cdot t(e)$ for $t \in T(H,k)$, $r \in R_k$, and $e \in E(H)$. Thus we can view $T(H,k)$ as a module over the ring R_k. It is easy to check that if $t, t_1, t_2 \in T^{=}(H,k)$, then $t_1 + t_2, rt \in T^{=}(H,k)$; hence $T^{=}(H,k)$ is a submodule of $T(H,k)$.

If H has an isthmus, then H^* has a loop, and thus $F(H,k) = 0 = C(H^*,k)$; from now on we assume that H has no isthmus.

Let us imagine that graphs H and H^* are drawn on a sphere. Let v^* be a vertex of graph H^* corresponding to a certain area among areas on which graph H divides the whole sphere. Let $e_1, \ldots, e_q$ be the edges bounding this area. For the definition of flows these edges were somehow oriented, so now we can speak of the edges $e_1, \ldots, e_q$ as oriented clock-wise and counterclock-wise (assuming that the "center of the clock" is at the vertex v^*). Let us define a flow t_{v^*} as the flow equal to $+1$ on clock-wise oriented edges, -1 on edges oriented in the opposite direction, and equal to 0 on the remaining edges (that is, different from $e_1, \ldots, e_q$). Clearly, the flow t_{v^*} is balanced.

To a given coloring s^* from $S(H^*, k)$, we associate the balanced flow:

$$t_{s^*} = \sum_{v^* \in V(H^*)} s^*(v^*) t_{v^*}. \tag{11}$$

Let us show that for each balanced flow t on H there exist exactly k colorings s^* such that

$$t = t_{s^*}. \tag{12}$$

First we prove that the number of such colorings cannot be greater than k. To this end we fix a vertex v_0 from $V(H^*)$ and show that a coloring s^* satisfying condition (12) can be uniquely determined by its values $s^*(v_0^*)$. Because of the connectivity of the graph H^* it is sufficient to show that the value $s^*(v_1^*)$ is uniquely determined where v_1^* is a vertex adjacent to v_0^*. Let l be the edge dual to the edge connecting v_0^* and v_1^*. According to (12) and (11)

$$t(l) = t_{s^*}(l) = \pm(s^*(v_0^*) - s^*(v_1^*)) \tag{13}$$

(The sign depends on the orientation of the edge l), and this relation allows us to determine $s^*(v_1^*)$ from $s^*(v_0^*)$ and t.)

Because $s^*(v_0^*)$ can assume at most k values, the number of colorings satisfying (12) cannot be greater than k. Let us now show that indeed all k cases can be implemented.

Let us fix a spanning tree D^* of graph H^*. Let us take for the value of $s^*(v_0^*)$ an arbitrary element of the ring R_k and define values of s^* on other vertices of graph H^* according to the above described procedure using relation (13) only for edges dual to the edges of the tree D^*. Let us show that the resulting coloring will satisfy the equality $t(l) = t_{s^*}(l)$ for the other edges as well. These edges form a tree W. By construction of s^* the flow $t - t_{s^*}$ is equal to zero outside W, and thus its restriction to W is balanced as well. But the only balanced flow on a forest is the flow identically equal to zero.

In order to complete the proof of equality (4) it remains to note that a coloring is proper if and only if the flow corresponding to it is non-degenerate.

In order to pass from (5) to (6) it suffices to note that by definition $m(G) = m(G^*)$, by Euler's Theorem $n(G) - m(G) + n(G^*) = 1 + s$, and there is a natural one-to-one correspondence between the sets $\{H^* | H \leq G\}$ and $\{L | L \preceq G^*\}$: namely, H^* is obtained from G^* by contracting edges dual to edges from $V(G) \setminus V(H)$.

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