Approximation of Riemann's Zeta Function by Finite Dirichlet Series. II

Yu. V. Matiyasevich

Steklov Institute of Mathematics at St.Petersburg, Russia

http://logic.pdmi.ras.ru/~yumat/personaljournal/finitedirichlet

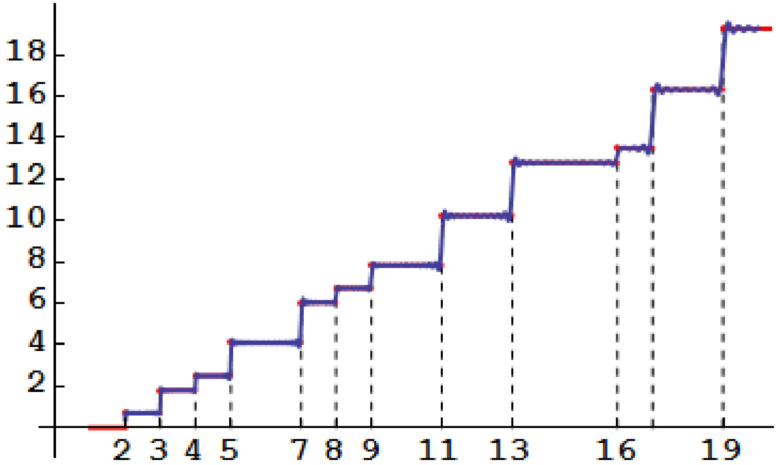
Zeta zeroes are very knowledgable

Zeta zeroes "know about"

- prime numbers (via von Mangoldt's theorem)
- the initial trivial zeroes
- other non-trivial zeroes
- the pole of the zeta functionvia the zeroes of the factor $1 - 2 \cdot 2^{-s}$ cancelling the pole

Theorem of von Mangoldt

$$\psi(x) \qquad x - \sum_{\substack{\zeta(\rho)=0 \\ |\rho|<400}} \frac{x^\rho}{\rho} - \ln(2\pi)$$



Approximations by Dirichlet series

$$\Delta_N(s) = 1^{-s} + \delta_{N,2} 2^{-s} + \cdots + \delta_{N,N} N^{-s}$$

$$N = 2M + 1$$

$$\Delta_N(\overline{\rho_M}) = \cdots = \Delta_N(\overline{\rho_1}) = 0 = \Delta_N(\rho_1) = \cdots = \Delta(\rho_M)$$

$$\begin{vmatrix} 1 & 1 & \dots & 1 & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ n^{-\overline{\rho_1}} & n^{-\rho_1} & \dots & n^{-\overline{\rho_M}} & n^{-\rho_M} & n^{-s} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ N^{-\overline{\rho_1}} & N^{-\rho_1} & \dots & N^{-\overline{\rho_M}} & N^{-\rho_M} & N^{-s} \end{vmatrix} = \sum_{n=1}^{N} \tilde{\delta}_{N,n} n^{-s}$$

Approximations by Dirichlet series

$$\begin{vmatrix} 1 & 1 & \dots & 1 & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ n^{-\overline{\rho_1}} & n^{-\rho_1} & \dots & n^{-\overline{\rho_M}} & n^{-\rho_M} & n^{-s} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ N^{-\overline{\rho_1}} & N^{-\rho_1} & \dots & N^{-\overline{\rho_M}} & N^{-\rho_M} & N^{-s} \end{vmatrix} = \sum_{n=1}^{N} \tilde{\delta}_{N,n} n^{-s}$$

$$\tilde{\delta}_{N,n} = (-1)^{n+1} \begin{vmatrix} 1 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (n-1)^{-\overline{\rho_1}} & (n-1)^{-\rho_1} & \dots & (n-1)^{-\overline{\rho_M}} & (n-1)^{-\rho_M} \\ (n+1)^{-\overline{\rho_1}} & (n+1)^{-\rho_1} & \dots & (n+1)^{-\overline{\rho_M}} & (n+1)^{-\rho_M} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ N^{-\overline{\rho_1}} & N^{-\rho_1} & \dots & N^{-\overline{\rho_M}} & N^{-\rho_M} \end{vmatrix}$$

$$\delta_{N,n} = \frac{\tilde{\delta}_{N,n}}{\tilde{\delta}_{N,1}} \qquad \Delta_N(s) = \sum_{n=1}^{N} \delta_{N,n} n^{-s}$$

Trivial zeroes for $M = 1550$, $N = 2M + 1 = 3101$

$$
\begin{aligned}
0 &= \Delta_N(-2 - 1.884\ldots \cdot 10^{-1510}) \\
0 &= \Delta_N(-4 + 2.013\ldots \cdot 10^{-1504}) \\
0 &= \Delta_N(-6 - 1.158\ldots \cdot 10^{-1498}) \\
0 &= \Delta_N(-8 + 4.508\ldots \cdot 10^{-1493}) \\
0 &= \Delta_N(-10 - 1.316\ldots \cdot 10^{-1487}) \\
0 &= \Delta_N(-12 + 3.066\ldots \cdot 10^{-1482}) \\
0 &= \Delta_N(-14 - 5.931\ldots \cdot 10^{-1477}) \\
0 &= \Delta_N(-16 + 9.796\ldots \cdot 10^{-1472}) \\
0 &= \Delta_N(-18 - 1.410\ldots \cdot 10^{-1466}) \\
0 &= \Delta_N(-20 + 1.797\ldots \cdot 10^{-1461}) \\
0 &= \Delta_N(-22 - 2.054\ldots \cdot 10^{-1456}) \\
0 &= \Delta_N(-24 + 2.126\ldots \cdot 10^{-1451})
\end{aligned}
$$

Non-trivial zeroes for $M = 1550$, $N = 2M + 1 = 3101$

$$
\begin{aligned}
0 &= \Delta_N(\rho_{M+1} - 5.154\ldots \cdot 10^{-1157} + 1.120\ldots \cdot 10^{-1156}\mathrm{i}) \\
0 &= \Delta_N(\rho_{M+201} - 4.922\ldots \cdot 10^{-890} - 9.995\ldots \cdot 10^{-891}\mathrm{i}) \\
0 &= \Delta_N(\rho_{M+401} - 3.159\ldots \cdot 10^{-735} - 2.750\ldots \cdot 10^{-735}\mathrm{i}) \\
0 &= \Delta_N(\rho_{M+601} + 8.765\ldots \cdot 10^{-619} + 4.575\ldots \cdot 10^{-618}\mathrm{i}) \\
0 &= \Delta_N(\rho_{M+801} + 2.075\ldots \cdot 10^{-524} + 1.197\ldots \cdot 10^{-524}\mathrm{i}) \\
0 &= \Delta_N(\rho_{M+1001} + 1.980\ldots \cdot 10^{-447} - 3.397\ldots \cdot 10^{-448}\mathrm{i}) \\
0 &= \Delta_N(\rho_{M+1201} - 1.034\ldots \cdot 10^{-381} - 1.354\ldots \cdot 10^{-382}\mathrm{i}) \\
0 &= \Delta_N(\rho_{M+1401} + 1.466\ldots \cdot 10^{-326} - 1.835\ldots \cdot 10^{-326}\mathrm{i}) \\
0 &= \Delta_N(\rho_{M+1601} + 2.281\ldots \cdot 10^{-278} - 3.603\ldots \cdot 10^{-278}\mathrm{i}) \\
0 &= \Delta_N(\rho_{M+1801} - 7.799\ldots \cdot 10^{-237} - 3.726\ldots \cdot 10^{-237}\mathrm{i}) \\
0 &= \Delta_N(\rho_{M+2001} + 5.921\ldots \cdot 10^{-201} - 6.855\ldots \cdot 10^{-201}\mathrm{i}) \\
0 &= \Delta_N(\rho_{M+2201} + 8.049\ldots \cdot 10^{-170} + 1.359\ldots \cdot 10^{-169}\mathrm{i}) \\
0 &= \Delta_N(\rho_{M+2401} - 2.001\ldots \cdot 10^{-142} - 7.023\ldots \cdot 10^{-142}\mathrm{i})
\end{aligned}
$$

Extra zeroes for $M = 1550$, $N = 2M + 1 = 3101$

$$\Delta_N(s) \leftrightharpoons \eta(s) \quad = \quad 1^{-s} - 2^{-s} + 3^{-s} - 4^{-s} + \cdots = (1 - 2 \cdot 2^{-s})\zeta(s)$$

$$1 - 2 \cdot 2^{-s} = 0 \quad \Longleftrightarrow \quad s = s_k = 1 + \frac{2\pi k}{\ln(2)}\mathrm{i}, \quad k = 0, \pm 1, \pm 2, \ldots$$

$$\begin{aligned}
0 &= \Delta_N(s_{50} - 5.481\ldots \cdot 10^{-133} - 5.546\ldots \cdot 10^{-133}\mathrm{i}) \\
0 &= \Delta_N(s_{100} - 1.109\ldots \cdot 10^{-132} - 1.306\ldots \cdot 10^{-134}\mathrm{i}) \\
0 &= \Delta_N(s_{150} - 5.743\ldots \cdot 10^{-133} + 5.543\ldots \cdot 10^{-133}\mathrm{i}) \\
0 &= \Delta_N(s_{200} - 6.157\ldots \cdot 10^{-136} + 2.613\ldots \cdot 10^{-134}\mathrm{i}) \\
0 &= \Delta_N(s_{250} - 5.220\ldots \cdot 10^{-133} - 5.537\ldots \cdot 10^{-133}\mathrm{i}) \\
0 &= \Delta_N(s_{300} - 1.108\ldots \cdot 10^{-132} - 3.917\ldots \cdot 10^{-134}\mathrm{i}) \\
0 &= \Delta_N(s_{350} - 6.004\ldots \cdot 10^{-133} + 5.528\ldots \cdot 10^{-133}\mathrm{i}) \\
0 &= \Delta_N(s_{400} - 2.461\ldots \cdot 10^{-135} + 5.220\ldots \cdot 10^{-134}\mathrm{i})
\end{aligned}$$

Notation

$$\mu_{N,n} = \sum_{m|n} \mu\left(\frac{n}{m}\right) \delta_{N,m}$$

$$\mu(k) = \begin{cases} (-1)^m, & \text{if } k \text{ is the product of } m \text{ different primes} \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned}
\mu_{N,1} &= 1 \\
\mu_{N,2} &= \delta_{N,2} - 1 \\
\mu_{N,3} &= \delta_{N,3} - 1 \\
\mu_{N,4} &= \delta_{N,4} - \delta_{N,2} \\
\mu_{N,5} &= \delta_{N,5} - 1 \\
\mu_{N,6} &= \delta_{N,6} - \delta_{N,3} - \delta_{N,2} + 1 \\
\mu_{N,7} &= \delta_{N,7} - 1 \\
\mu_{N,8} &= \delta_{N,8} - \delta_{N,4} \\
\mu_{N,9} &= \delta_{N,9} - \delta_{N,3}
\end{aligned}$$

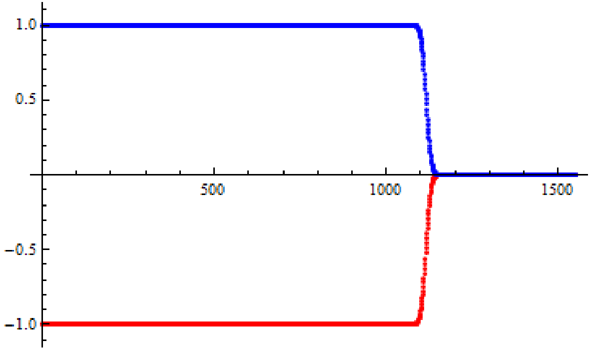
Case $N = 2001$

$$\begin{aligned}
\mu_{2001,2} &= -2 + 9.93613\ldots \cdot 10^{-86} \\
\mu_{2001,3} &= -1.49042\ldots \cdot 10^{-85} \\
\mu_{2001,4} &= -1.35392\ldots \cdot 10^{-199} \\
\mu_{2001,5} &= +4.28708\ldots \cdot 10^{-297} \\
\mu_{2001,6} &= -1.39904\ldots \cdot 10^{-377} \\
\mu_{2001,7} &= -8.46908\ldots \cdot 10^{-444} \\
\mu_{2001,8} &= -3.00897\ldots \cdot 10^{-499} \\
\mu_{2001,9} &= +2.56119\ldots \cdot 10^{-546} \\
\mu_{2001,10} &= +9.47153\ldots \cdot 10^{-587} \\
\mu_{2001,11} &= -2.22088\ldots \cdot 10^{-622} \\
\mu_{2001,12} &= +1.65346\ldots \cdot 10^{-653} \\
\mu_{2001,13} &= -1.33219\ldots \cdot 10^{-680} \\
\mu_{2001,14} &= -2.89063\ldots \cdot 10^{-705} \\
\mu_{2001,15} &= -2.27283\ldots \cdot 10^{-726}
\end{aligned}$$

Case $N = 2001$

$$
\begin{aligned}
\delta_{N,3} - \delta_{N,1} &= -1.49042\ldots \cdot 10^{-85} \\
\delta_{N,4} - \delta_{N,2} &= -1.35392\ldots \cdot 10^{-199} \\
\delta_{N,5} - \delta_{N,1} &= +4.28708\ldots \cdot 10^{-297} \\
\delta_{N,6} - \delta_{N,3} - \delta_{N,2} + \delta_{N,1} &= -1.39904\ldots \cdot 10^{-377} \\
\delta_{N,7} - \delta_{N,1} &= -8.46908\ldots \cdot 10^{-444} \\
\delta_{N,8} - \delta_{N,4} &= -3.00897\ldots \cdot 10^{-499} \\
\delta_{N,9} - \delta_{N,3} &= +2.56119\ldots \cdot 10^{-546} \\
\delta_{N,10} - \delta_{N,5} - \delta_{N,2} + \delta_{N,1} &= +9.47153\ldots \cdot 10^{-587} \\
\delta_{N,11} - \delta_{N,1} &= -2.22088\ldots \cdot 10^{-622} \\
\delta_{N,12} - \delta_{N,6} - \delta_{N,4} + \delta_{N,2} &= +1.65346\ldots \cdot 10^{-653} \\
\delta_{N,13} - \delta_{N,1} &= -1.33219\ldots \cdot 10^{-680} \\
\delta_{N,14} - \delta_{N,7} - \delta_{N,2} + \delta_{N,1} &= -2.89063\ldots \cdot 10^{-705} \\
\delta_{N,15} - \delta_{N,5} - \delta_{N,3} + \delta_{N,3} &= -2.27283\ldots \cdot 10^{-726}
\end{aligned}
$$

Coefficients $\delta_{3101,n}$, red for even n, blue for odd n



$\delta_{3101,n}$

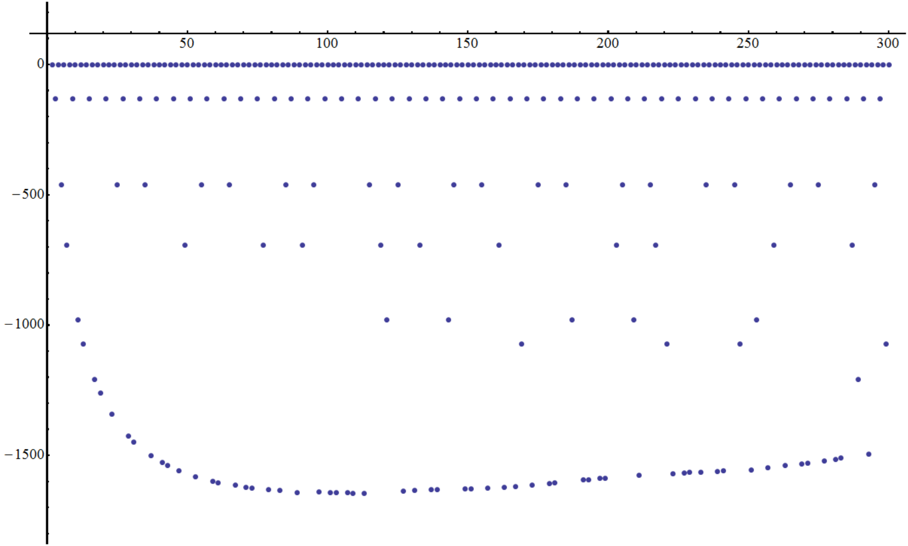
$$\delta_{3101,n} - 1$$

$$|\delta_{3101,n} - 1|$$

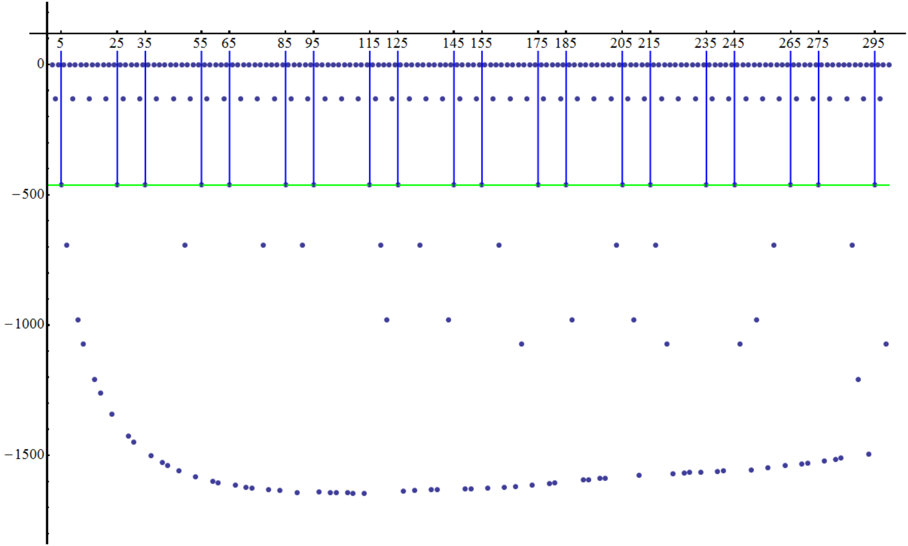
$\log_{10}|\delta_{3101,n} - 1|$

Plot of $\log_{10}|\delta_{3101,n} - 1|$

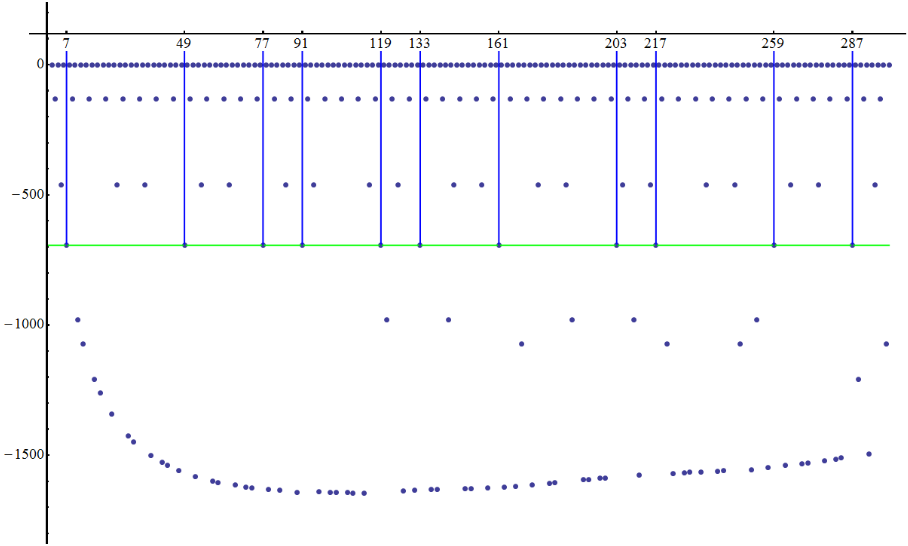
Plot of $\log_{10} |\delta_{3101,n} - 1|$



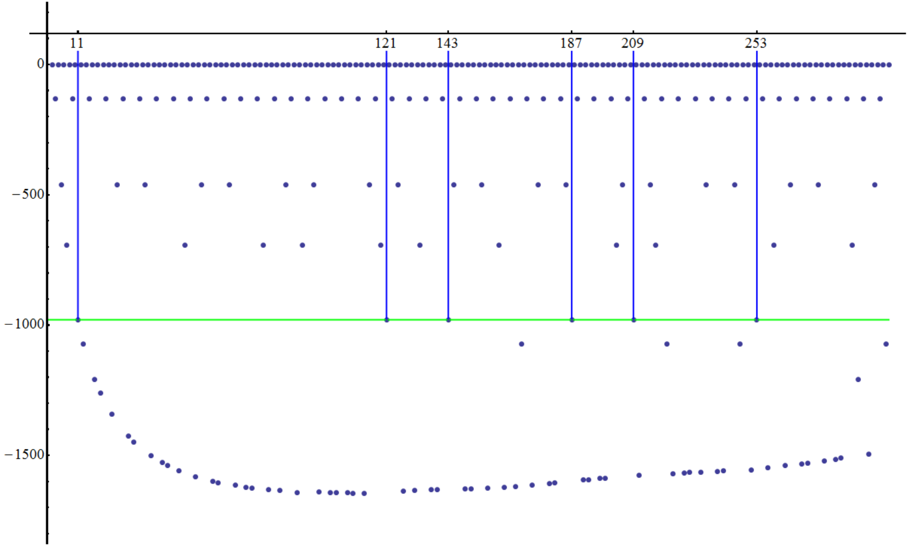
Plot of $\log_{10}|\delta_{3101,n} - 1|$



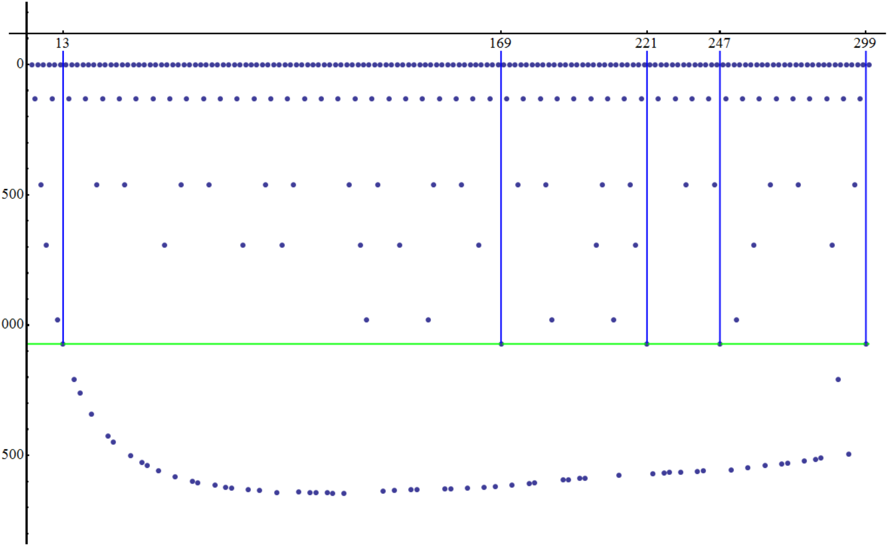
Plot of $\log_{10} |\delta_{3101,n} - 1|$



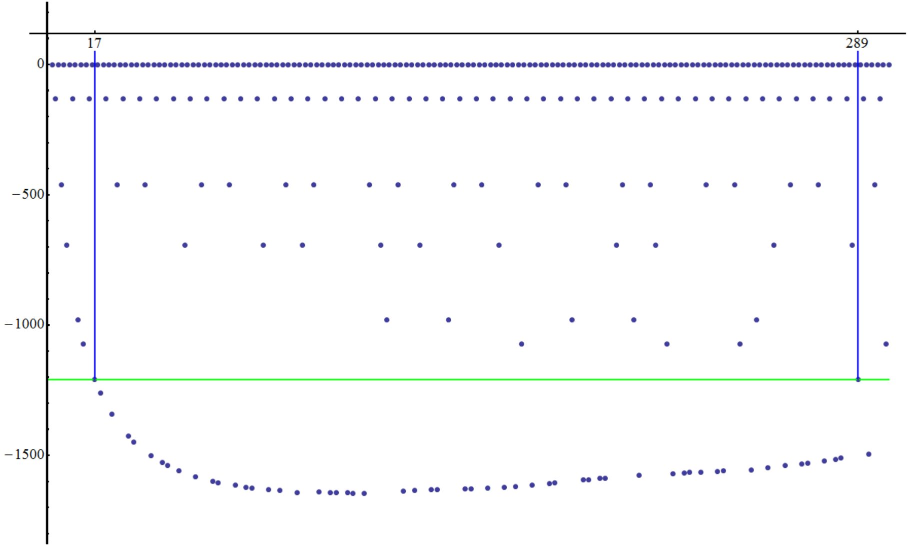
Plot of $\log_{10}|\delta_{3101,n} - 1|$



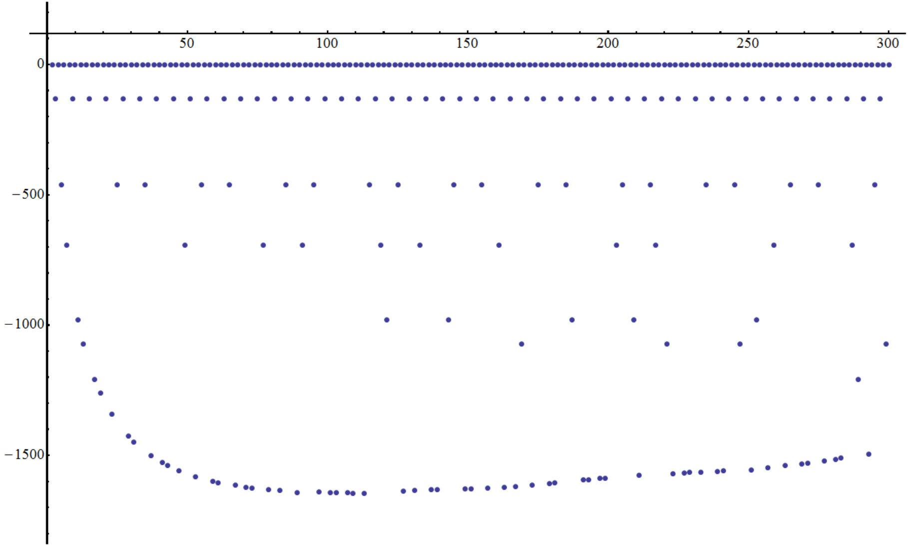
Plot of $\log_{10}|\delta_{3101,n} - 1|$



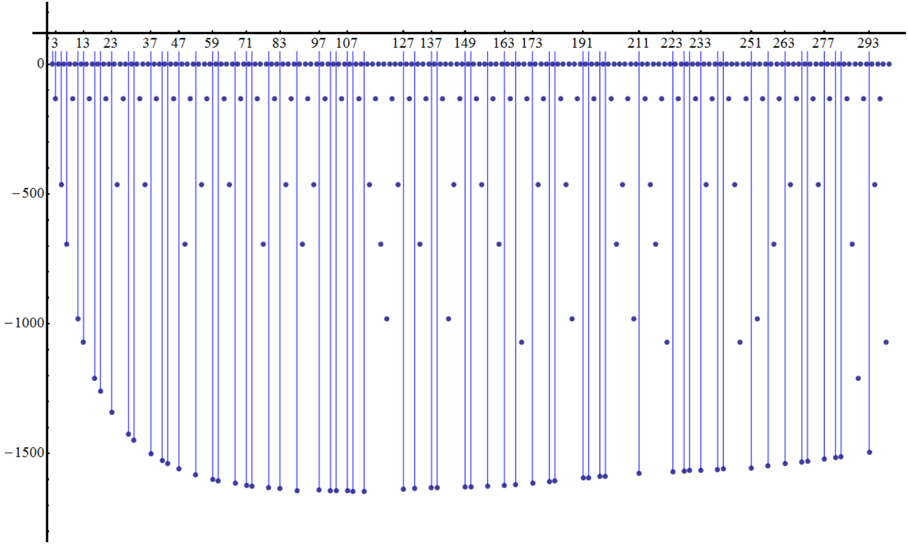
Plot of $\log_{10} |\delta_{3101,n} - 1|$



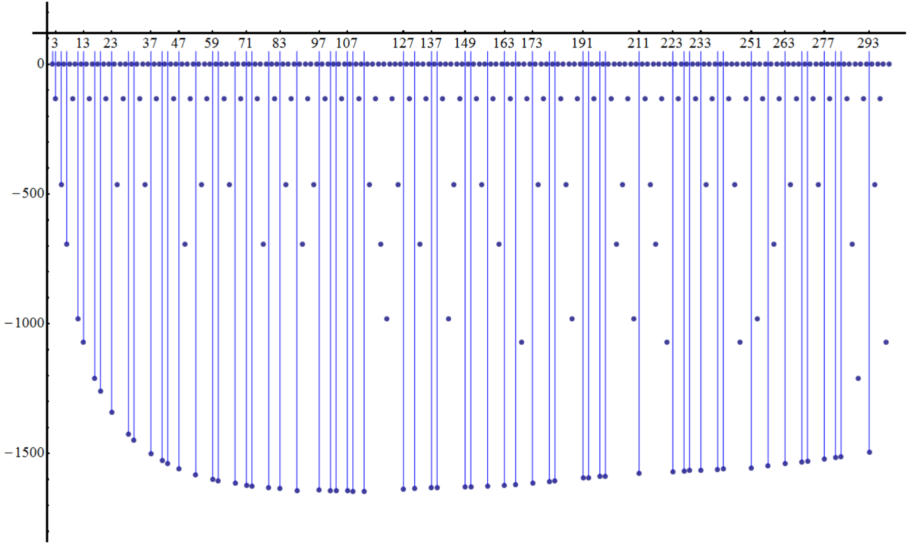
Plot of $\log_{10} |\delta_{3101,n} - 1|$



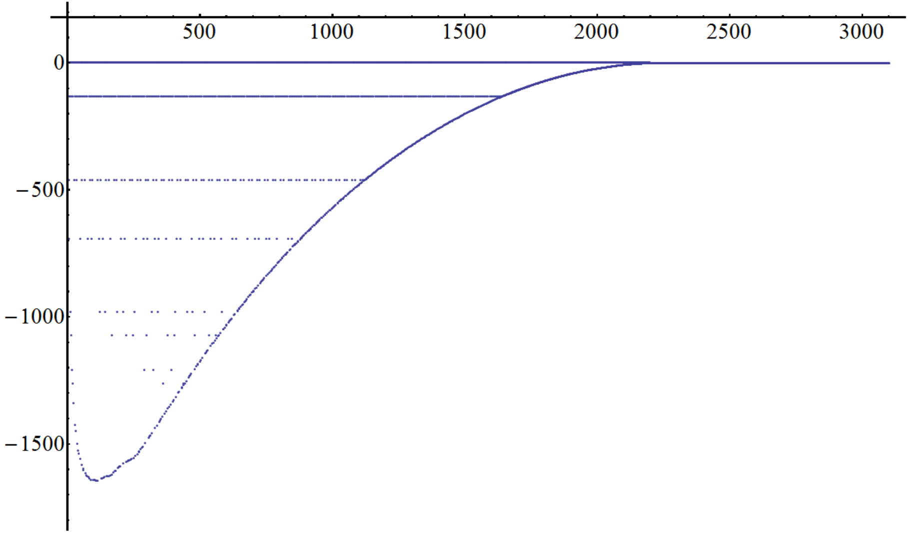
Plot of $\log_{10}|\delta_{3101,n} - 1|$



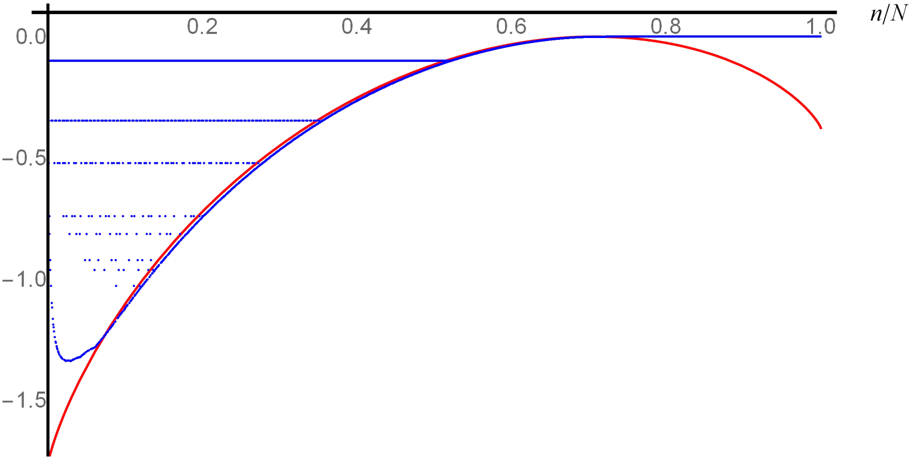
Plot of $\log_{10} |\delta_{3101,n} - 1| =$ Sieve of Eratosthenes



Plot of $\log_{10}|\delta_{3101,n} - 1|$

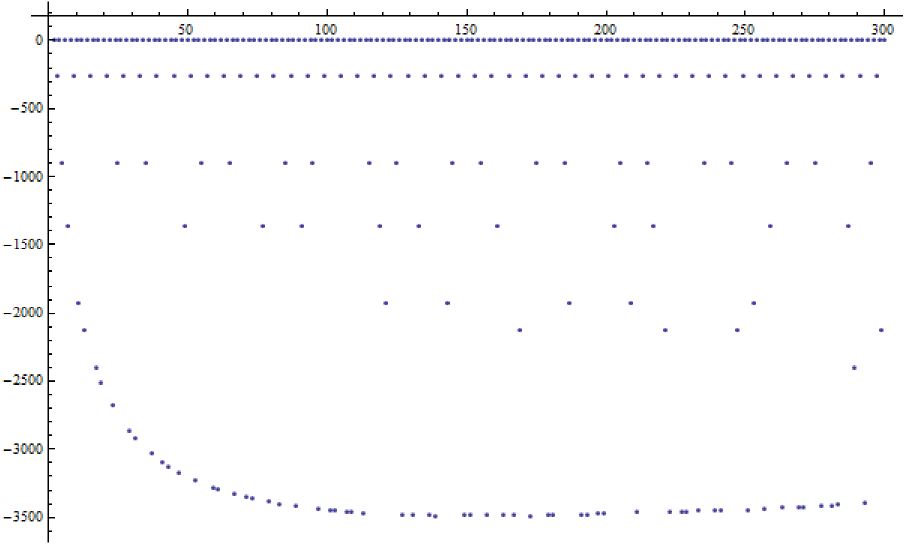


Scaled Plot of $\ln|\delta_{N,n} - (-1)^{n+1}|/N$ for $N = 6001$

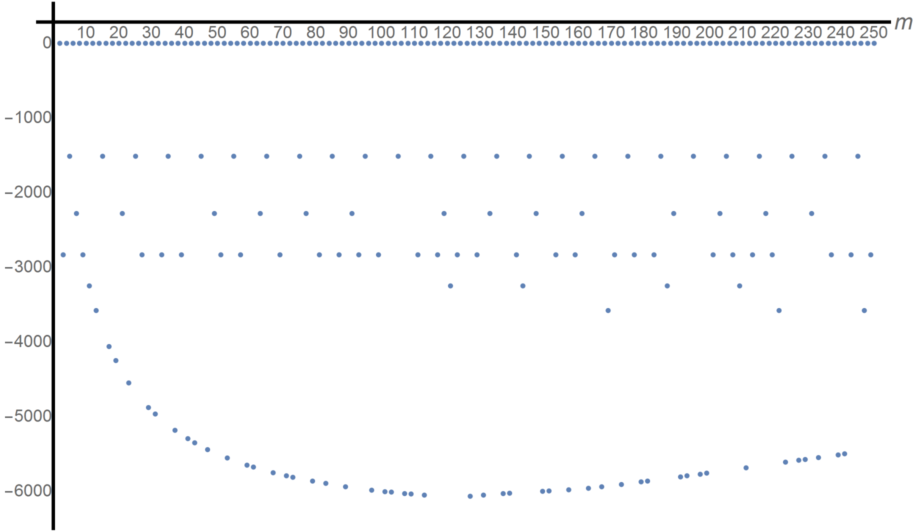


$\ell(a) = (a-1)\ln(1-a) - 2a\ln(a) + (a+1)\ln(a+1) - \ln(2\sqrt{2}+3)$

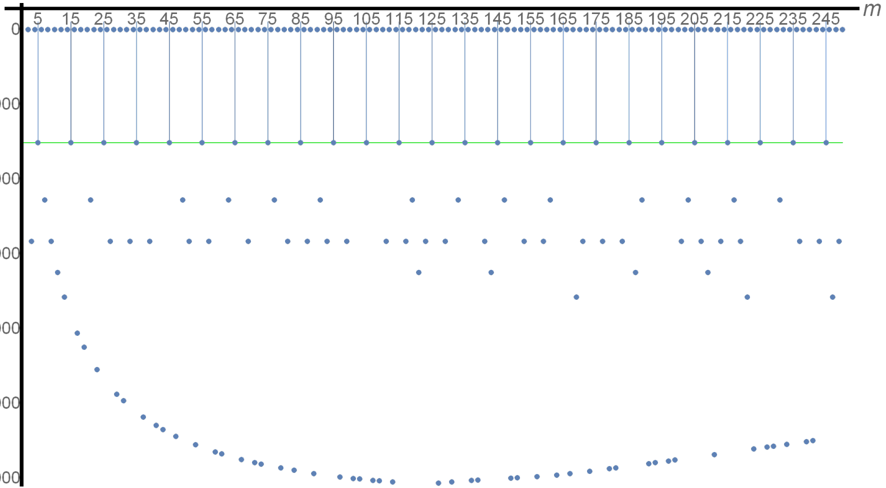
Plot of $\log_{10}|\delta_{6001,n} - 1|$ (repeated)



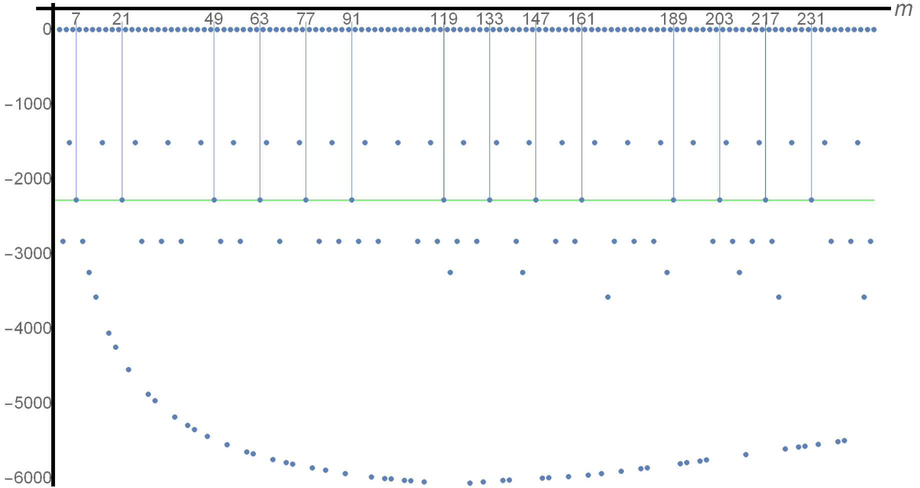
Finer Structure: Plot of $\log_{10} |\delta_{10001,3m} - \delta_{10001,3}|$



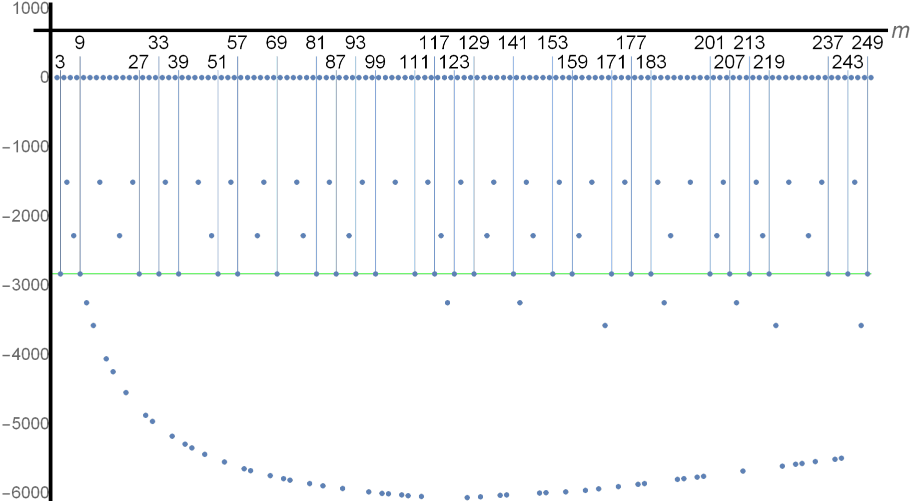
Finer Structure: Plot of $\log_{10}|\delta_{10001,3m} - \delta_{10001,3}|$



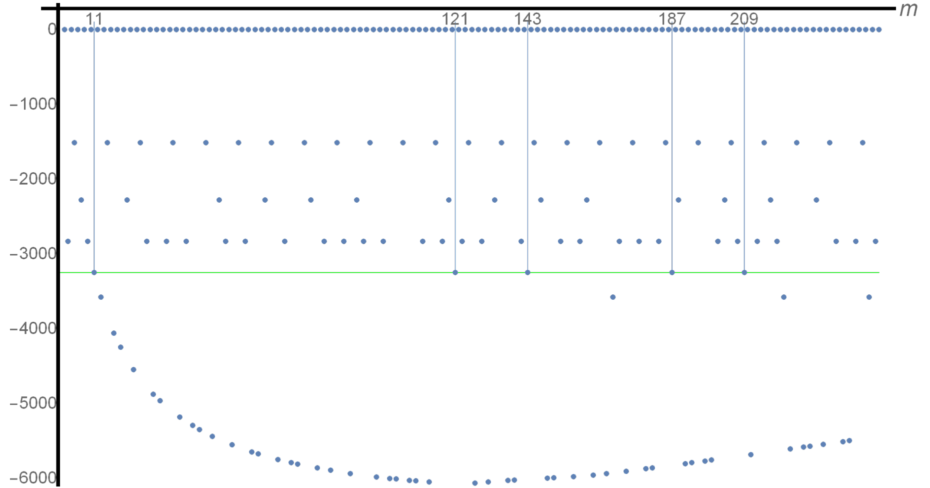
Finer Structure: Plot of $\log_{10}|\delta_{10001,3m} - \delta_{10001,3}|$



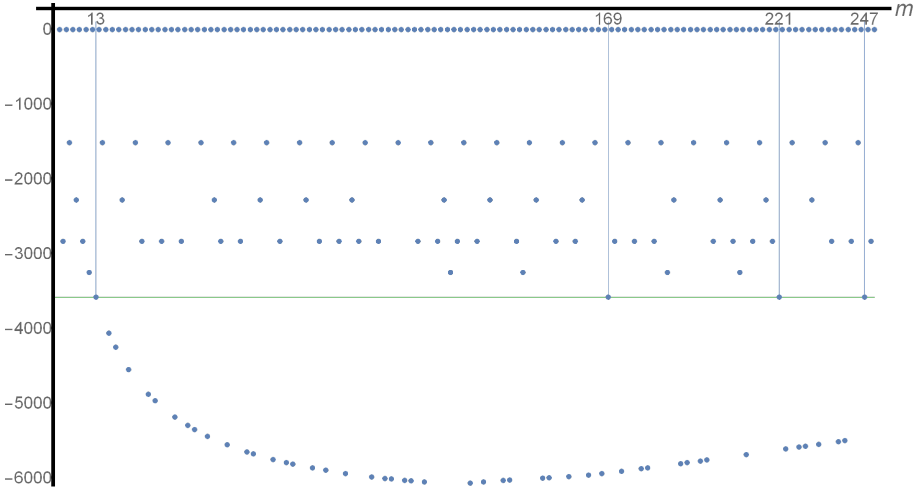
Finer Structure: Plot of $\log_{10} |\delta_{10001,3m} - \delta_{10001,3}|$



Finer Structure: Plot of $\log_{10} |\delta_{10001,3m} - \delta_{10001,3}|$

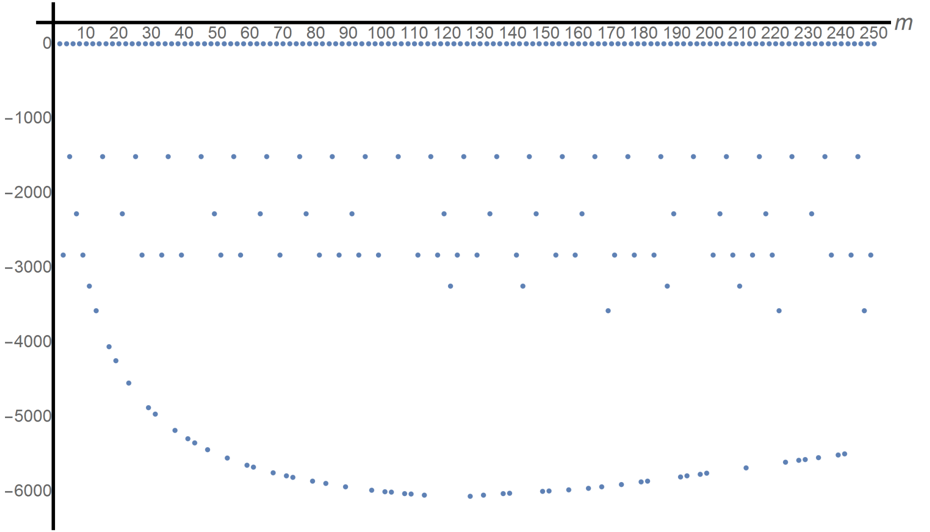


Finer Structure: Plot of $\log_{10} |\delta_{10001,3m} - \delta_{10001,3}|$



Finer Structure: Plot of $\log_{10}|\delta_{10001,3m} - \delta_{10001,3}|$

$= \text{Eratosthenes Sieve with primes order } 2, 5, 7, 3, 11, 13, \ldots$



Expected Fractal Structure

Let n range over the arithmetical progression $d, 2d, \ldots, md, \ldots$ with

$$d = 2^{k_2} 3^{k_3} 5^{k_5} \ldots$$

Corresponding Eratosthenes sublevel splits according to the divisibility of m by p_1, p_2, ... where these prime numbers are ordered in such a way that

$$p_1^{k_{p_1}+1} < p_2^{k_{p_2}+1} < \cdots < p_j^{k_{p_j}+1} < \ldots$$

In the previous example $m = 3$, hence $k_2 = 0$, $k_3 = 1$, $k_5 = k_7 = \cdots = 0$ and $p_1 = 2,\ p_2 = 5,\ p_3 = 7,\ p_4 = 3,\ p_5 = 11,\ p_6 = 13,\ \ldots$ according to

$$2^1 < 5^1 < 7^1 < 3^2 < 11^1 < 13^2 < \ldots$$

Zeroes from Euler Product

$$\begin{aligned}
\zeta(s) &= 1^{-s} + 2^{-s} + \cdots + k^{-s} + \dots \\
&= \prod_{p \text{ is prime}} \frac{1}{1 - p^{-s}}
\end{aligned}$$

$$\begin{aligned}
\zeta_2(s) &= \prod_{\substack{p \text{ is prime} \\ p \neq 2}} \frac{1}{1 - p^{-s}} \\
&= 1^{-s} + 3^{-s} + \cdots + (2k+1)^{-s} + \dots \\
&= L(2, \chi_1, s) \\
&= (1 - 2^{-s})\zeta(s)
\end{aligned}$$

An Example

$$\zeta_2(s) \quad = \quad (1-2^{-s})\zeta(s)$$

Let us take 100 (pairs of conjugate) zeros of the zeta function,

$$\frac{1}{2} \pm \mathrm{i}\gamma_k, \qquad k = 1, \dots, 100,$$

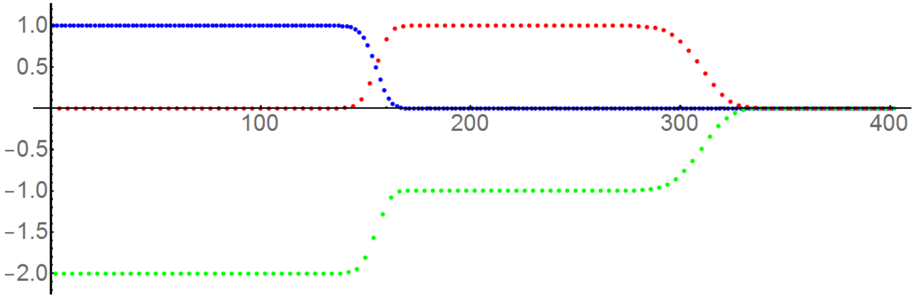
and 201 zeros of $1-2^{-s}$,

$$0, \pm\frac{2\pi \mathrm{i}}{\ln(2)}, \dots, \pm 100\frac{2\pi \mathrm{i}}{\ln(2)}$$

calculate corresponding 402 determinants of size 401 and normalize them getting numbers

$$\delta_{2,200,402,1}, \dots, \delta_{2,200,402,402}$$

$\delta_{2,200,402,n}$ for $\zeta_2(s) = (1-2^{-s})\zeta(s)$



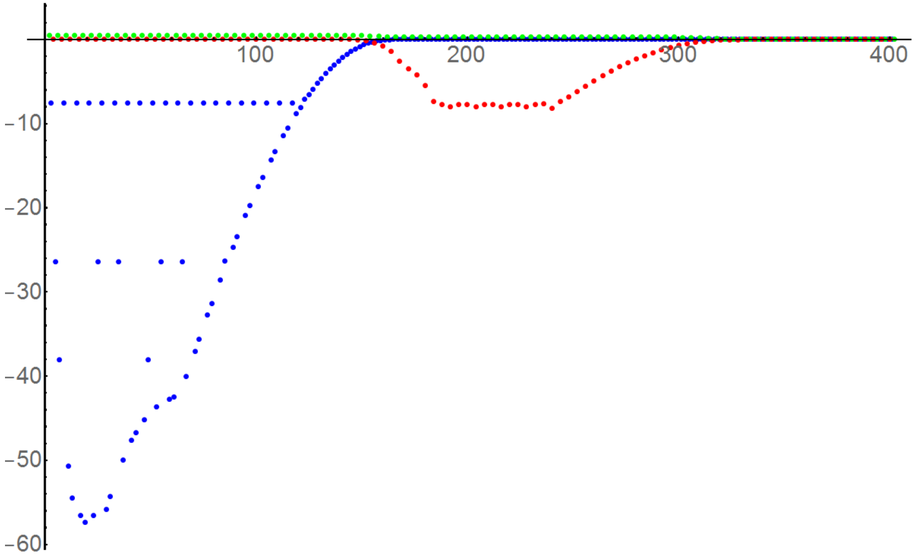
Blue for $n \equiv 1 \pmod 2$, Red for $n \equiv 0 \pmod 4$, Green for $n \equiv 2 \pmod 4$

$n < 120 \Rightarrow$ the δ's are close to the coefficients of $(1-2^{-s})(1-2\cdot 2^{-s})\zeta(s)$

$190 < n < 290 \Rightarrow$ the δ's are close to the coefficients of $-2^{-s}(1-2\cdot 2^{-s})\zeta(s)$

$330 < n \Rightarrow$ the δ's are very small

$\log_{10} |\delta_{2,200,402,n} - 1|$ for $\zeta_2(s) = (1 - 2^{-s})\zeta(s)$



Blue for $n \equiv 1 \pmod 2$, Red for $n \equiv 0 \pmod 4$, Green for $n \equiv 2 \pmod 4$

Theorem of von Mangoldt

Теорема (Hans Carl Fridrich von Mangoldt [1895]).

$$\psi(x) = x - \sum_{\xi(\rho)=0} \frac{x^\rho}{\rho} - \sum_n \frac{x^{-2n}}{-2n} - \ln(2\pi)$$

$$\begin{aligned}
\zeta(s) &= \prod_{p \text{ – простое}} \frac{1}{1-p^{-s}} \\
\xi(s) &= \pi^{-\frac{s}{2}}(s-1)\Gamma\left(\frac{s}{2}+1\right)\zeta(s) \\
&= \xi(0) \prod_{\xi(\rho)=0} \left(1 - \frac{s}{\rho}\right)
\end{aligned}$$

Davenport–Heilbronn function

$$f(s) = \sum_{n=1}^{\infty} d(n) n^{-s}$$

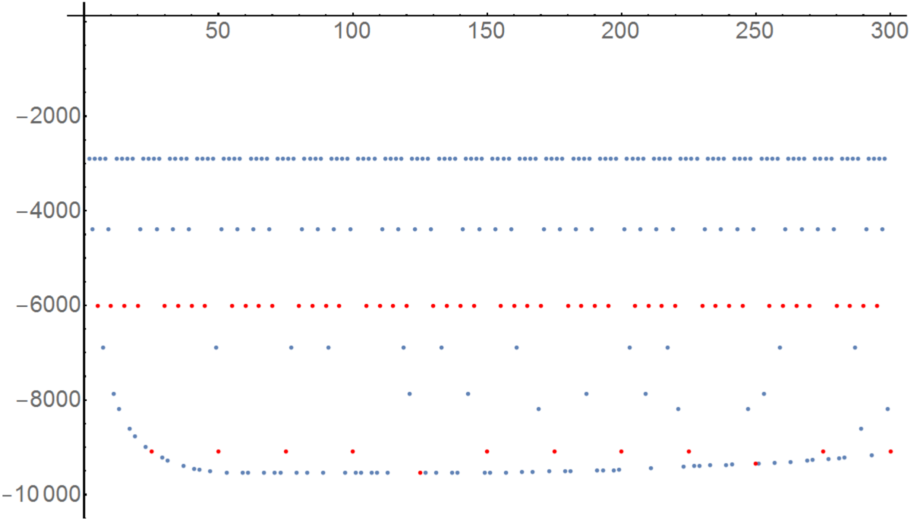
where

$$d(n) = \begin{cases} 0, & \text{if } n \equiv 0 \pmod 5 \\ 1, & \text{if } n \equiv 1 \pmod 5 \\ \tau, & \text{if } n \equiv 2 \pmod 5 \\ -\tau, & \text{if } n \equiv 3 \pmod 5 \\ -1, & \text{if } n \equiv 4 \pmod 5 \end{cases}$$

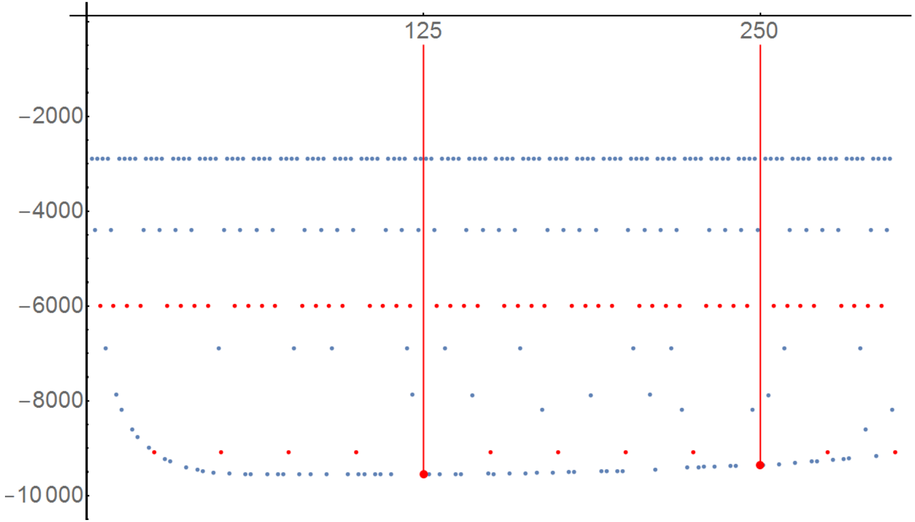
and

$$\tau = \frac{-2 + \sqrt{10 - 2\sqrt{5}}}{-1 + \sqrt{5}}$$

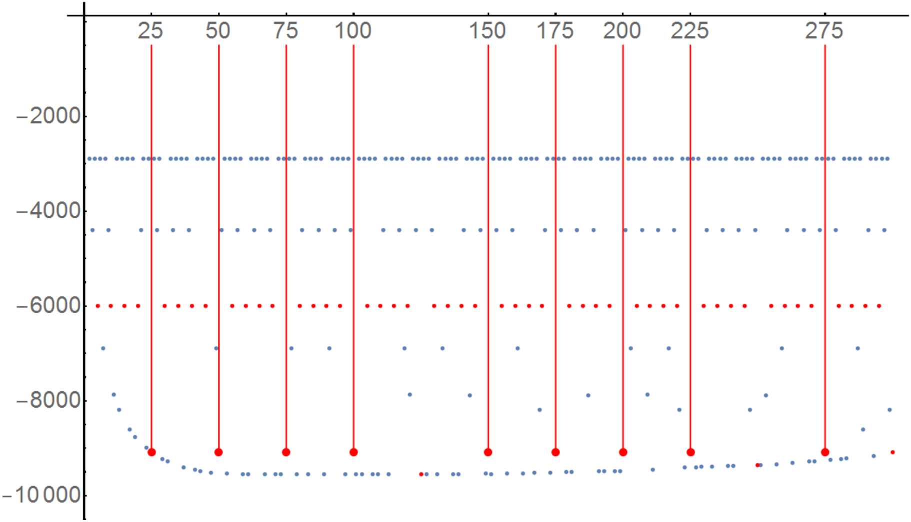
Sieve of Eratosthenes for $f(s)$, red whenever $5|n$



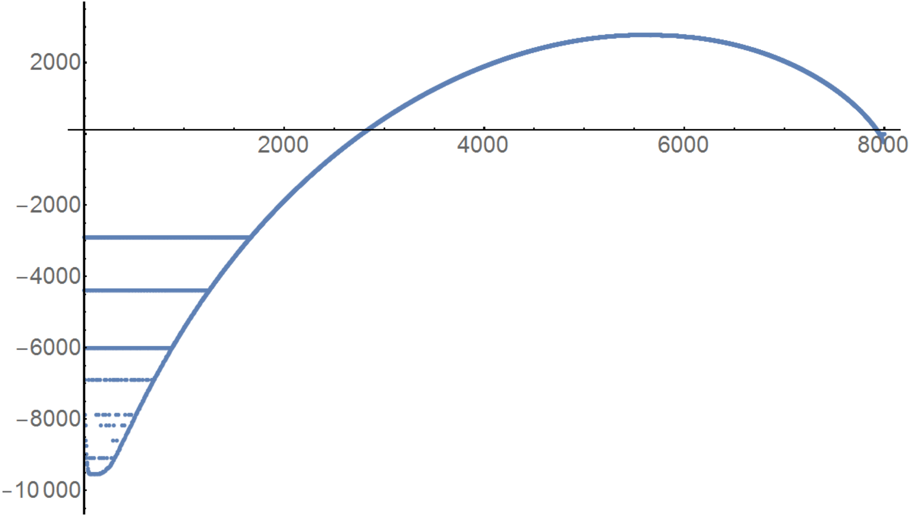
Sieve of Eratosthenes for $f(s)$



Sieve of Eratosthenes for $f(s)$



Sieve of Eratosthenes for $f(s)$



Plot of $\log_{10}|\delta^{(f)}_{7999,n} - d(n)|$

Approximation of the zeta function

$$\Delta_N(s) = \sum_{n=1}^{N} \delta_{N,n} n^{-s}$$
$$\leftrightharpoons \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s}$$
$$= (1 - 2 \cdot 2^{-s})\zeta(s)$$

$$\zeta(s) \stackrel{?}{\approx} \frac{\Delta_N(s)}{1 - 2 \cdot 2^{-s}}$$

s	$\left\lvert \frac{\Delta_{3001}(s)}{1-2\cdot 2^{-s}} - \zeta(s) \right\rvert$
25	$4.2671\ldots \cdot 10^{-135}$
2	$3.9256\ldots \cdot 10^{-128}$
1000i	$4.4184\ldots \cdot 10^{-128}$
$\frac{1}{2} + 10\mathrm{i}$	$1.0953\ldots \cdot 10^{-127}$
$-1 + 100\mathrm{i}$	$3.6324\ldots \cdot 10^{-127}$
-25	$1.6415\ldots \cdot 10^{-126}$
$2 + 1000\mathrm{i}$	$2.3063\ldots \cdot 10^{-125}$
$\frac{1}{2} + 1000\mathrm{i}$	$3.9630\ldots \cdot 10^{-124}$
$-1 + 1000\mathrm{i}$	$1.4867\ldots \cdot 10^{-118}$
$-10 + 1000\mathrm{i}$	$8.2377\ldots \cdot 10^{-103}$
$\frac{1}{2} + 5000\mathrm{i}$	$6.5116\ldots \cdot 10^{-64}$

Function $\nu_{N,L}(s)$

$$\Delta_N(s) = \sum_{n=1}^{N} \delta_{N,n} n^{-s} \leftrightharpoons \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s} = (1 - 2 \cdot 2^{-s})\zeta(s)$$

$$\nu_N(s) = \frac{\Delta_N(s)}{\zeta(s)} = \frac{\sum_{n=1}^{N} \delta_{N,n} n^{-s}}{\sum_{n=1}^{\infty} n^{-s}} = \sum_{n=1}^{\infty} \mu_{N,n} n^{-s} \leftrightharpoons 1 - 2 \cdot 2^{-s}$$

$$\mu_{N,n} = \sum_{m|n} \mu\left(\frac{n}{m}\right) \delta_{N,n} \qquad \delta_{N,n} = 0, \text{ if } n > N$$

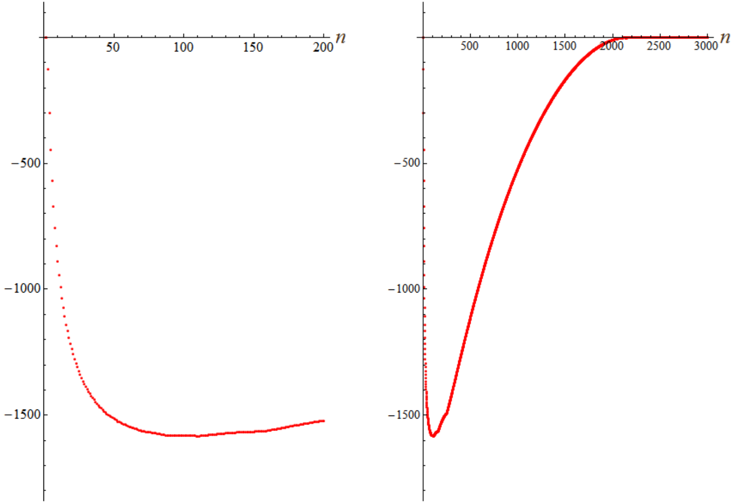
$$\zeta(s) = \frac{\Delta_N(s)}{\nu_N(s)} \qquad \nu_{N,L}(s) = \sum_{n=1}^{L} \mu_{N,n} n^{-s}$$

$$\zeta(s) \overset{?}{\approx} \frac{\Delta_N(s)}{\nu_{N,L}(s)} = \frac{\sum_{n=1}^{N} \delta_{N,n} n^{-s}}{\sum_{n=1}^{L} \mu_{N,n} n^{-s}}$$

Calculation of $\zeta(s)$ at $s = \frac{1}{4} + 1000i$ for $N = 3001$

L	$\mu_{N,L}$	$\left\lvert \zeta(s) - \frac{\Delta(s)}{\nu_{N,L}(s)} \right\rvert$
2	$-2 + 1.43\ldots \cdot 10^{-127}$	$2.24128\ldots \cdot 10^{-127}$
3	$-2.14787\ldots \cdot 10^{-127}$	$1.57968\ldots \cdot 10^{-299}$
4	$-1.62673\ldots \cdot 10^{-299}$	$4.85859\ldots \cdot 10^{-448}$
5	$+5.29034\ldots \cdot 10^{-448}$	$1.00748\ldots \cdot 10^{-569}$
6	$-1.14817\ldots \cdot 10^{-569}$	$1.83153\ldots \cdot 10^{-672}$
7	$+2.16930\ldots \cdot 10^{-672}$	$3.15150\ldots \cdot 10^{-756}$
8	$-3.85941\ldots \cdot 10^{-756}$	$2.34266\ldots \cdot 10^{-829}$
9	$-2.95462\ldots \cdot 10^{-829}$	$3.17791\ldots \cdot 10^{-891}$
10	$+4.11503\ldots \cdot 10^{-891}$	$6.45307\ldots \cdot 10^{-946}$
11	$-8.55748\ldots \cdot 10^{-946}$	$6.55682\ldots \cdot 10^{-994}$
12	$+8.88627\ldots \cdot 10^{-994}$	$1.00011\ldots \cdot 10^{-1036}$
13	$+1.38282\ldots \cdot 10^{-1036}$	$2.32048\ldots \cdot 10^{-1074}$
14	$-3.26844\ldots \cdot 10^{-1074}$	$1.18994\ldots \cdot 10^{-1108}$

Numbers $\log_{10}|\mu_{3001,n}|$



$$\mu_{N,n} = \sum_{m|n} \mu\left(\frac{n}{m}\right) \delta_{N,n}$$