Approximation of Riemann's Zeta Function by Finite Dirichlet Series. III

Yu. V. Matiyasevich

Steklov Institute of Mathematics at St.Petersburg, Russia

http://logic.pdmi.ras.ru/~yumat/personaljournal/finitedirichlet

The functional equation for the zeta function

$$\zeta(s) = 1^{-s} + 2^{-s} + \cdots + n^{-s} \dots$$

$$\xi(s) = g(s)\zeta(s) \qquad g(s) = \pi^{-\frac{s}{2}}(s-1)\Gamma\left(\frac{s}{2}+1\right)$$

$$\xi(s) = \xi(1-s)$$

$$\Delta_N(s) = 1^{-s} + \delta_{N,2}2^{-s} + \cdots + \delta_{N,N}N^{-s}$$

$$g(s)\Delta_N(s) \stackrel{?}{=} g(1-s)\Delta_N(1-s)$$

Dummy Functional Equation

$$\xi(s) = g(s)\zeta(s) \qquad g(s) = \pi^{-\frac{s}{2}}(s-1)\Gamma\left(\frac{s}{2}+1\right) \qquad \xi(s) = \xi(1-s)$$

$$\begin{aligned}
\xi(s) = \quad & g(s)\zeta(s) && = \sum_{n=1}^{\infty} g(s)n^{-s} \\
= \quad & g(1-s)\zeta(1-s) && = \sum_{n=1}^{\infty} g(1-s)n^{s-1} \\
& && = \sum_{n=1}^{\infty} \frac{g(s)n^{-s} + g(1-s)n^{s-1}}{2} \\
& && = \sum_{n=1}^{\infty} f_n(s)
\end{aligned}$$

$$f_n(s) = \frac{g(s)n^{-s} + g(1-s)n^{s-1}}{2} = f_n(1-s)$$

New Finite Dirichlet Series

$$\xi(s) = \xi(1-s) = \sum_{n=1}^{\infty} f_n(s) \qquad f_n(s) = \frac{g(s)n^{-s} + g(1-s)n^{s-1}}{2} = f_n(1-s)$$

$$\cdots = \zeta(\overline{\rho_3}) = \zeta(\overline{\rho_2}) = \zeta(\overline{\rho_1}) = 0 = \zeta(\rho_1) = \zeta(\rho_2) = \zeta(\rho_3) = \dots$$

$$\tilde{\Delta}_N^{\Gamma}(s) = \begin{vmatrix} f_1(\rho_1) & \dots & f_1(\rho_{N-1}) & f_1(s) \\ \vdots & \ddots & \vdots & \vdots \\ f_N(\rho_1) & \dots & f_N(\rho_{N-1}) & f_N(s) \end{vmatrix} = \sum_{n=1}^{N} \tilde{\delta}_{N,n}^{\Gamma} f_n(s)$$

$$\delta_{N,n}^{\Gamma} = \frac{\tilde{\delta}_{N,n}^{\Gamma}}{\tilde{\delta}_{N,1}^{\Gamma}} \qquad \Delta_N^{\Gamma}(s) = \sum_{n=1}^{N} \delta_{N,n}^{\Gamma} n^{-s}$$

$$\frac{\tilde{\Delta}_{N,n}^{\Gamma}(s)}{\tilde{\delta}_{N,1}^{\Gamma}} = \frac{g(s)\Delta_{N,n}^{\Gamma}(s) + g(1-s)\Delta_{N,n}^{\Gamma}(1-s)}{2}$$

Non-trivial zeroes for $M = 1000$, $N = 2M + 1 = 2001$

$$
\begin{aligned}
0 &= \Delta_N^\Gamma(\rho_1 - 1.064\ldots \cdot 10^{-394} - 2.365\ldots \cdot 10^{-395}\mathrm{i}) \\
0 &= \Delta_N^\Gamma(\rho_{201} + 4.279\ldots \cdot 10^{-395} - 2.773\ldots \cdot 10^{-394}\mathrm{i}) \\
0 &= \Delta_N^\Gamma(\rho_{401} - 7.344\ldots \cdot 10^{-395} + 1.315\ldots \cdot 10^{-394}\mathrm{i}) \\
0 &= \Delta_N^\Gamma(\rho_{601} - 9.371\ldots \cdot 10^{-395} - 6.435\ldots \cdot 10^{-395}\mathrm{i}) \\
0 &= \Delta_N^\Gamma(\rho_{801} - 8.177\ldots \cdot 10^{-395} - 1.095\ldots \cdot 10^{-394}\mathrm{i}) \\
0 &= \Delta_N^\Gamma(\rho_{1001} - 8.978\ldots \cdot 10^{-395} - 8.220\ldots \cdot 10^{-395}\mathrm{i}) \\
0 &= \Delta_N^\Gamma(\rho_{1201} + 1.398\ldots \cdot 10^{-394} - 3.255\ldots \cdot 10^{-394}\mathrm{i}) \\
0 &= \Delta_N^\Gamma(\rho_{1401} - 9.902\ldots \cdot 10^{-395} - 2.367\ldots \cdot 10^{-395}\mathrm{i}) \\
0 &= \Delta_N^\Gamma(\rho_{1601} - 9.706\ldots \cdot 10^{-395} - 4.342\ldots \cdot 10^{-395}\mathrm{i}) \\
0 &= \Delta_N^\Gamma(\rho_{1801} - 9.370\ldots \cdot 10^{-395} + 6.438\ldots \cdot 10^{-395}\mathrm{i}) \\
0 &= \Delta_N^\Gamma(\rho_{2001} + 2.075\ldots \cdot 10^{-323} + 1.276\ldots \cdot 10^{-322}\mathrm{i})
\end{aligned}
$$

Function $\nu^{\Gamma}_{N,L}(s)$

$$\Delta^{\Gamma}_{N}(s) = \sum_{n=1}^{N} \delta^{\Gamma}_{N,n} n^{-s} \leftrightharpoons \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s} = (1 - 2 \cdot 2^{-s})\zeta(s)$$

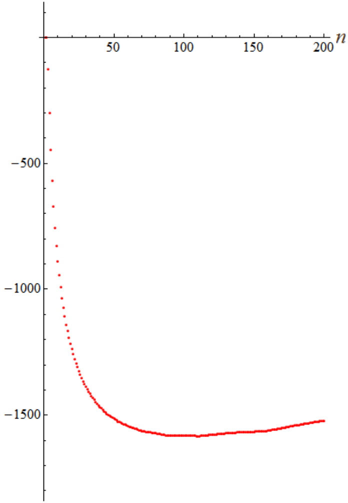
$$\nu^{\Gamma}_{N}(s) = \frac{\Delta^{\Gamma}_{N}(s)}{\zeta(s)} = \frac{\sum_{n=1}^{N} \delta^{\Gamma}_{N,n} n^{-s}}{\sum_{n=1}^{\infty} n^{-s}} = \sum_{n=1}^{\infty} \mu^{\Gamma}_{N,n} n^{-s} \leftrightharpoons 1 - 2 \cdot 2^{-s}$$

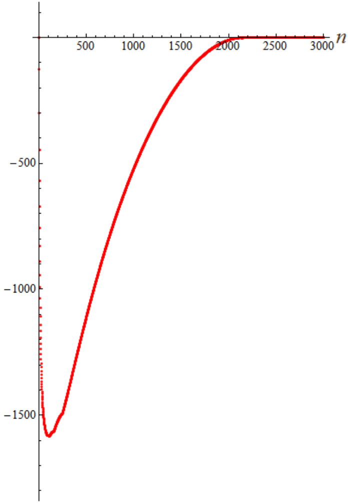
$$\mu^{\Gamma}_{N,n} = \sum_{m|n} \mu\left(\frac{n}{m}\right) \delta^{\Gamma}_{N,n} \qquad \delta^{\Gamma}_{N,n} = 0, \text{ if } n > N$$

$$\zeta(s) = \frac{\Delta^{\Gamma}_{N}(s)}{\nu^{\Gamma}_{N}(s)(s)} \qquad \nu^{\Gamma}_{N,L}(s) = \sum_{n=1}^{L} \mu^{\Gamma}_{N,n} n^{-s}$$

$$\zeta(s) \overset{?}{\approx} \frac{\Delta^{\Gamma}_{N}(s)}{\nu^{\Gamma}_{N,L}(s)} = \frac{\sum_{n=1}^{N} \delta^{\Gamma}_{N,n} n^{-s}}{\sum_{n=1}^{L} \mu^{\Gamma}_{N,n} n^{-s}}$$

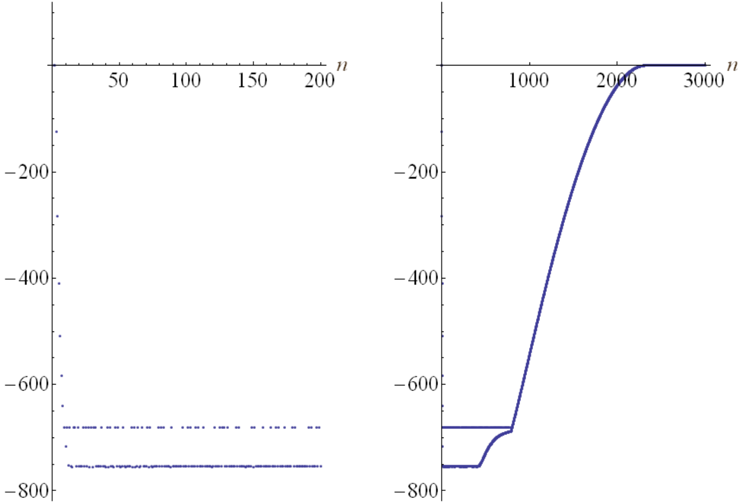
Numbers $\log_{10}|\mu_{3001,n}|$ (slide repeated)





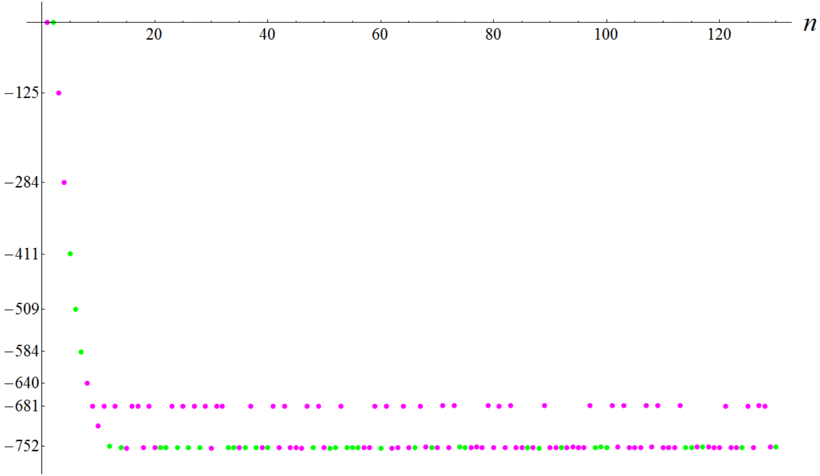
$$\mu_{N,n} = \sum_{m|n} \mu\left(\frac{n}{m}\right)\delta_{N,n}$$

Numbers $\log_{10}|\mu^{\Gamma}_{3000,n}|$

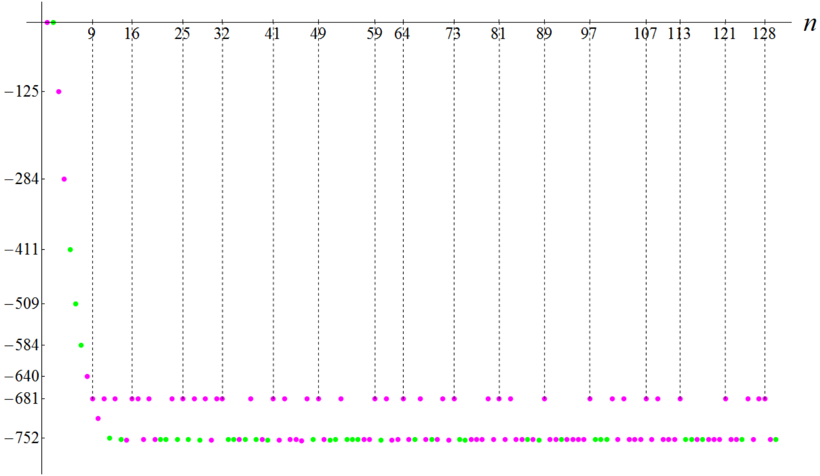


$$\mu^{\Gamma}_{N,n} = \sum_{m|n} \mu\left(\frac{n}{m}\right) \delta^{\Gamma}_{N,n}$$

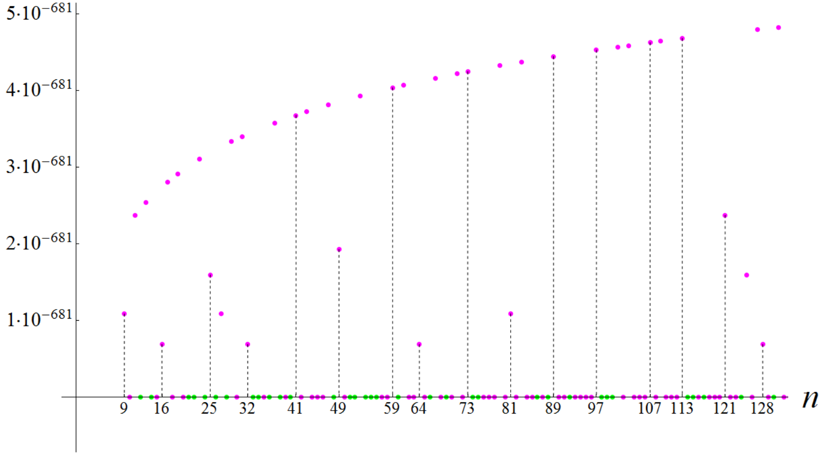
$\log_{10}(|\mu^{\Gamma}_{3000,n}|)$, magenta, if $\mu^{\Gamma}_{3000,n} > 0$, green otherwise



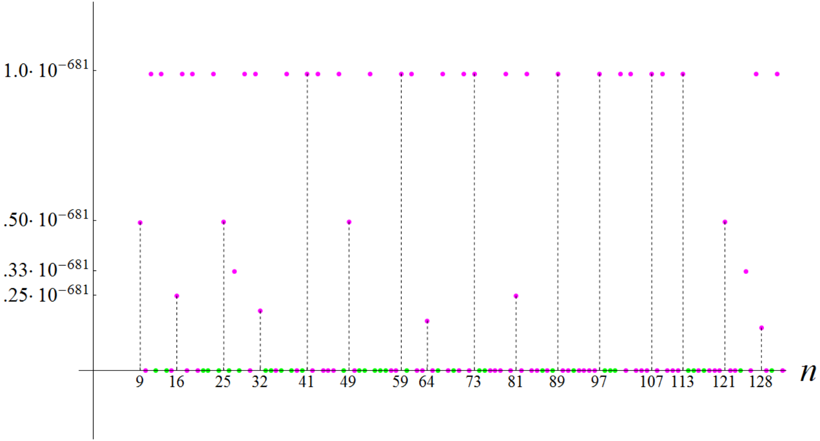
$\log_{10}(|\mu^{\Gamma}_{3000,n}|)$, magenta, if $\mu^{\Gamma}_{3000,n} > 0$, green otherwise

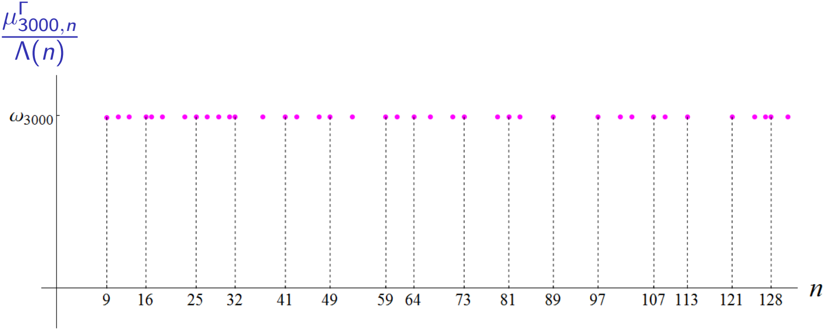


$\mu^{\Gamma}_{3000,n}$, magenta, if $\mu^{\Gamma}_{3000,n} > 0$, green otherwise



$\frac{\mu^{\Gamma}_{3000,n}}{\ln(n)}$, magenta, if $\mu^{\Gamma}_{3000,n} > 0$, green otherwise

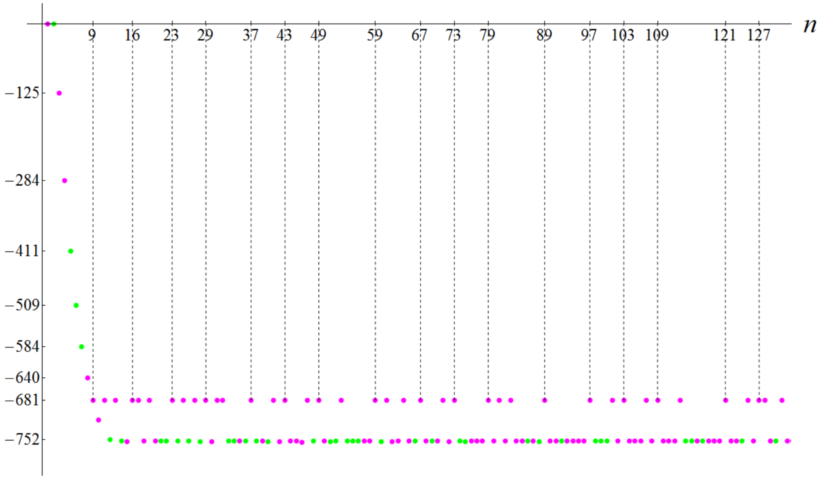




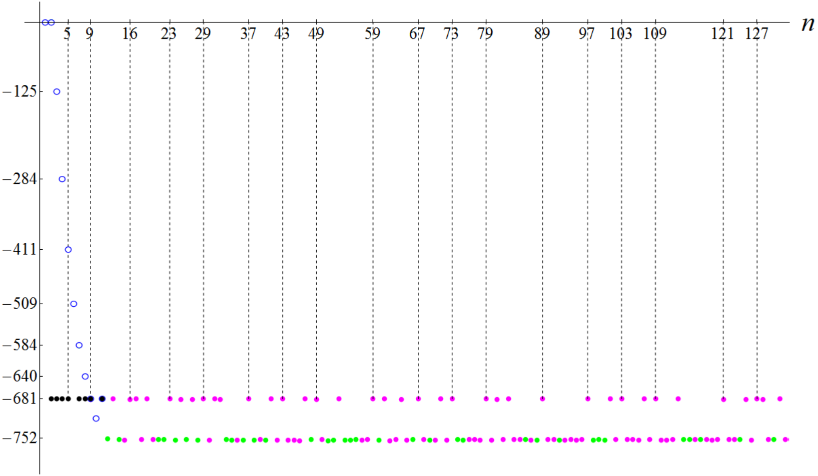
$$\omega_{3000} = \frac{\mu^{\Gamma}_{3000,13}}{\ln(13)} = 9.895811\ldots \cdot 10^{-682}$$

$$\left| \frac{\mu^{\Gamma}_{3000,p^k} / \ln(p)}{\omega_{3000}} - 1 \right| < 3.85\ldots \cdot 10^{-73} \text{ for } 13 \le p^k \le 419,\ \ p \text{ is a prime}$$

$$\left| \frac{\mu^{\Gamma}_{3000,p^k}}{\ln(p)} - \omega_{3000} \right| < 3.81\ldots \cdot 10^{-754} \text{ for } 13 \le p^k \le 419,\ \ p \text{ is a prime}$$



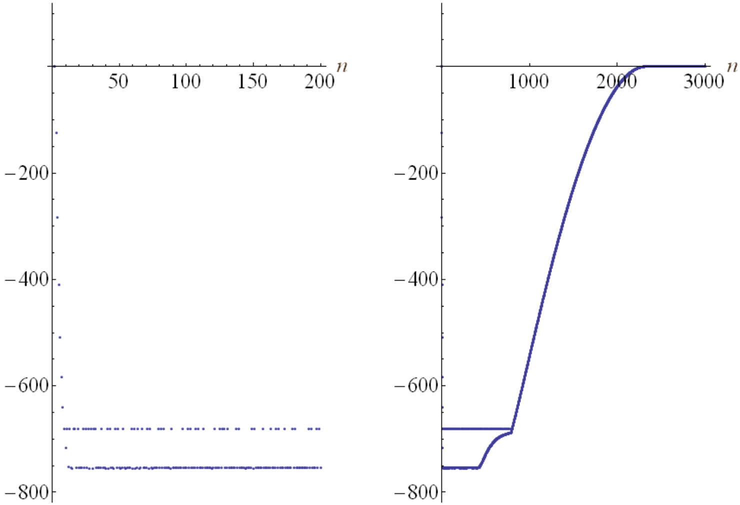
$$\frac{\Delta^{\Gamma}_{3000}(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \mu^{\Gamma}_{3000,n} n^{-s}$$



$$\sum_{n=1}^{11} \left(\omega_{3000}\Lambda(n) - \mu^{\Gamma}_{3000,n}\right) n^{-s} + \frac{\Delta^{\Gamma}_{3000}(s)}{\zeta(s)} \quad \leftrightharpoons \quad \omega_{3000} \sum_{n=1}^{\infty} \Lambda(n) n^{-s}$$

$$\sum_{n=1}^{\infty} \Lambda(n) n^{-s} \quad = \quad -\frac{\zeta'(s)}{\zeta(s)}$$

Numbers $\log_{10}|\mu^{\Gamma}_{3000,n}|$



$$-\omega_{3000}\sum_{n=1}^{\infty}\Lambda(n)n^{-s} \leftrightharpoons \sum_{n=1}^{11}\left(\omega_{3000}\Lambda(n)-\mu^{\Gamma}_{3000,n}\right)n^{-s}+\frac{\Delta^{\Gamma}_{3000}(s)}{\zeta(s)} \quad \overset{?}{\approx} -\omega_{3000}\frac{\zeta'(s)}{\zeta(s)}$$

Calculating zeta derivative at zeros

$$\sum_{n=1}^{11} \left(\omega_{3000}\Lambda(n) - \mu^{\Gamma}_{3000,n}\right) n^{-s} + \frac{\Delta^{\Gamma}_{3000}(s)}{\zeta(s)} \quad \overset{?}{\approx} \quad -\omega_{3000}\frac{\zeta'(s)}{\zeta(s)}$$

$$\zeta(s)\sum_{n=1}^{11} \left(\omega_{3000}\Lambda(n) - \mu^{\Gamma}_{3000,n}\right) n^{-s} + \Delta^{\Gamma}_{3000}(s) \quad \overset{?}{\approx} \quad -\omega_{3000}\zeta'(s)$$

$$\Delta^{\Gamma}_{3000}\left(\tfrac{1}{2} + \mathrm{i}\gamma_k\right) \quad \overset{?}{\approx} \quad -\omega_{3000}\zeta'\left(\tfrac{1}{2} + \mathrm{i}\gamma_k\right)$$

$$\left|\frac{\Delta^{\Gamma}_{3000}\left(\frac{1}{2} + \mathrm{i}\gamma_{100}\right)}{-\omega_{3000}\zeta'\left(\frac{1}{2} + \mathrm{i}\gamma_{100}\right)} - 1\right| = 1.024\ldots \cdot 10^{-36}$$

$$\left|\frac{\Delta^{\Gamma}_{3000}\left(\frac{1}{2} + \mathrm{i}\gamma_{500}\right)}{-\omega_{3000}\zeta'\left(\frac{1}{2} + \mathrm{i}\gamma_{500}\right)} - 1\right| = 2.786\ldots \cdot 10^{-74}$$

Calculating zeta derivative at other points

$$\zeta(s)\sum_{n=1}^{11}\left(\omega_{3000}\Lambda(n)-\mu^{\Gamma}_{3000,n}\right)n^{-s}+\Delta^{\Gamma}_{3000}(s) \quad \overset{?}{\approx} \quad -\omega_{3000}\zeta'(s)$$

$$s=\frac{1}{4}+1000\mathrm{i}$$

$$\left|\frac{\zeta(s)\sum_{n=1}^{11}\left(\omega_{3000}\Lambda(n)-\mu^{\Gamma}_{3000,n}\right)n^{-s}+\Delta^{\Gamma}_{3000}(s)}{-\omega_{3000}\zeta'(s)}-1\right|=6.44\ldots\cdot 10^{-73}$$

Calculating both zeta and its derivative. I

$$\zeta(s)\zeta(s)\sum_{n=1}^{11}\left(\omega_{3000}\Lambda(n)-\mu^{\Gamma}_{3000,n}\right)n^{-s}+\Delta^{\Gamma}_{3000}(s) \approx -\omega_{3000}\zeta'(s)\zeta'(s)$$

$$\zeta(s)\zeta(s)\sum_{n=1}^{11}\left(\omega_{3500}\Lambda(n)-\mu^{\Gamma}_{3500,n}\right)n^{-s}+\Delta^{\Gamma}_{3500}(s) \approx -\omega_{3500}\zeta'(s)\zeta'(s)$$

Solving this system for $s=\frac{1}{4}+1000\mathrm{i}$ produces 908 correct decimal digits for $\zeta(s)$ and 72 correct decimal digits for $\zeta'(s)$.

Calculating both zeta and its derivative. II

$$\zeta(s)\zeta(s)\sum_{n=1}^{11}\left(\omega_{3000}\Lambda(n)-\mu^{\Gamma}_{3000,n}\right)n^{-s}+\Delta^{\Gamma}_{3000}(s) \approx -\omega_{3000}\zeta'(s)\zeta'(s)$$

$$\zeta(1-s)\zeta(1-s)\sum_{n=1}^{11}\left(\omega_{3000}\Lambda(n)-\mu^{\Gamma}_{3000,n}\right)n^{s-1}+\Delta^{\Gamma}_{3000}(1-s) \approx -\omega_{3000}\zeta$$

$$g(s)\zeta(s)\zeta(s)=g(1-s)\zeta(1-s)\zeta(1-s)$$

$$g'(s)\zeta(s)\zeta(s)+g(s)\zeta'(s)\zeta'(s)=-g'(1-s)\zeta(1-s)\zeta(1-s)-g(1-s)\zeta'(1-s)$$

Solving this system for $s=\frac{1}{4}+1000\mathrm{i}$ produces 752 correct decimal digits for $\zeta(s)$ and 72 correct decimal digits for $\zeta'(s)$.

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