Word Equations, Fibonacci Numbers, and Hilbert's Tenth Problem

Yuri Matiyasevich
Steklov Institute of Mathematics at Saint-Petersburg
27 Fontanka, Saint-Petersburg, 191023, Russia
URL: http://logic.pdmi.ras.ru/~yumat

In his tenth problem D. Hilbert [1] asked for an algorithm for deciding whether an arbitrary Diophantine equation has a solution in integers. As a possible way to establish the undecidability of Hilbert's tenth problem A.A.Markov suggested to prove the undecidability of *word equations.* Any such equation (w.l.o.g, in the two-letter alphabet $B = \{0, 1\}$) can be easily reduced to a Diophantine equation using the fact that every 2×2 matrix with natural number elements and the determinant equal to zero can be represented in a unique way as the product of matrices $B_0 = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ and $B_1 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. This allows us to treat a quadruple of non-negative integers $\langle a_{11}, a_{12}, a_{21}, a_{22}\rangle$ such that $a_{11}a_{22} - a_{21}a_{12} = 1$ as a word in the alphabet B, the concatenation being just matrix multiplication.

This approach to Hilbert's tenth problem turned out to be fruitless: G. S. Makanin [2] found a decision procedure for word equations. Nevertheless, we could try to revive Markov's idea by considering a wider class of word equations.

Every word $X = \chi_n\chi_{n-1}\dots\chi_1$ in the alphabet B can be viewed as the number

$$x = \chi_n u_n + \chi_{n-1}u_{n-1} + \cdots + \chi_1 u_1 \tag{1}$$

written in positional system with weights of digits being the Fibonacci numbers $u_1 = 1$, $u_2 = 1$, $u_3 = 2$, $u_4 = 3$, $u_5 = 5$, … (rather than traditional 1, 2, 4, 8, 16, …). According to Zeckendorf's theorem, every natural number x can be represented in the form (4) with additional restrictions $\chi_{i+1}\chi_i = 0$ and $\chi_1 = 0$, and in unique way.

Every word $\alpha_{i_n}\alpha_{i_{n-1}}\dots\alpha_{i_1}$ in the infinite alphabet $A = \{\alpha_1, \alpha_2, \dots\}$ can be presented as a word in the alphabet B

$$10^{i_m}10^{i_{m-1}}\dots 10^{i_1} \tag{2}$$

satisfying the restrictions of Zeckendorf's theorem. Thus we get, via Fibonacci numbers, a natural one-to-one correspondence between words in the infinite alphabet A and natural numbers, the kth word under this enumeration will be denoted Z_k.

Now in order to be able to transform an arbitrary word equation into an equivalent Diophantine equation we need only to express the *concatenation relation* $Z_k = Z_{k_1}Z_{k_2}$ by Diophantine equation(s). For this goal it is more convenient to code a word X by a quadruple $\langle v, w, x, y\rangle$ where x is as in (1), and

$$v = u_n, \quad w = u_{n-1}, \quad y = \chi_n u_{n-1} + \chi_{n-1}u_{n-2} + \dots + \chi_1 u_0. \tag{3}$$

Thanks to the equality $\phi^m = \phi u_m + u_{m-1}$ where ϕ is the golden ratio $\frac{1+\sqrt{5}}{2}$, the number

$$\phi x + y = \chi_n\phi^n + \chi_{n-1}\phi^{n-1} + \dots + \chi_1\phi \tag{4}$$

is represented by the word X in the positional system with weights of digits being powers of the golden ratio 1, ϕ, ϕ^2, ϕ^3, ... Now the relation $Z_{\langle v,w,x,y\rangle} = Z_{\langle v_1,w_1,x_1,y_1\rangle}Z_{\langle v_2,w_2,x_2,y_2\rangle}$ is just the equality

$$\phi x + y = (\phi x_1 + y_1)(\phi v_2 + w_2) + \phi x_2 + y_2. \tag{5}$$

It still remains to distinguish those quadruples $\langle v, w, x, y\rangle$ which are codes of words. It is not difficult to check that numbers defined by (1) and (3) satisfy the following conditions:

$$(w^2 + vw - v^2)^2 = 1, \tag{6}$$

$$v \le x < v + w, \tag{7}$$

$$\phi - 2 < y - x/\phi < \phi - 1. \tag{8}$$

On the other hand, according to an old theorem of Wasteels [5] solutions of the equation (6) are exactly consecutive Fibonacci numbers. Thus for a fixed value of x, conditions (6)–(8) uniquely determine the values of v, w, and y.

Conditions (5)–(8) can be easily transformed into desired Diophantine equations.

Now we are capable to transform into Diophantine equations not only word equations but a broader class of conditions on words. In particular, condition $v_1 = v_2$ expresses the equality of the lengths of words $Z_{\langle v_1,w_1,x_1,y_1\rangle}$ and $Z_{\langle v_2,w_2,x_2,y_2\rangle}$.

Open question. *Is there an algorithm for deciding whether an arbitrary system of word equations and equalities of length of words has a solution?*

As we have seen, the undecidability of this problem would give a proof of the undecidability of Hilbert's tenth problem very different from known today (see, for example, [4]).

In the above arithmetization of words we had the restriction that a word cannot contain "11" and cannot end by "1". This drawback can be easily eliminated by the following modifications. Having an arbitrary word in the alphabet B, we at first transform it into a "restricted" word by replacing each "1" by"10" and each "0" by "00"; formally, representation (1) is replaced by

$$x = \chi_n u_{2n} + \chi_{n-1} u_{2(n-1)} + \cdots + \chi_1 u_2. \tag{9}$$

Condition (6) is strengthened to

$$w^2 + vw - v^2 = 1 \tag{10}$$

to imply that $v = u_m$ with even m and hence the word $Z_{\langle v,w,x,y\rangle}$ has even length. In order to pass back from numbers to words we cut $Z_{\langle v,w,x,y\rangle}$ into blocks of length 2 and then replace "00" by "0" while both "10" and "01" are replaced by "1". Now instead of one-to-one correspondence between words and numbers we have one-to-many correspondence but this isn't an obstacle for transforming systems of word equations and length equalities into equivalent Diophantine equations.

Besides equalities of lengths of words, we cam impose conditions in the form "the length of the word $Z_{\langle v_1,w_1,x_1,y_1\rangle}$ is divisible by the length of the word $Z_{\langle v_2,w_2,x_2,y_2\rangle}$" (see [3]).

References

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