Word Equations, Fibonacci Numbers, and Hilbert's Tenth Problem

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David Hilbert, *Mathematical Problems* [1900]

10. Entscheidung der Lösbarkeit einer diophantischen Gleichung. Eine diophantische Gleichung mit irgendwelchen Unbekannten und mit ganzen rationalen Zahlkoefficienten sei vorgelegt: *man soll ein Verfahren angeben, nach welchen sich mittels einer endlichen Anzahl von Operationen entscheiden lässt, ob die Gleichung in ganzen rationalen Zahlen lösbar ist*.

10. Determination of the Solvability of a Diophantine Equation. Given a diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: *To devise a process according to which it can be determined by a finite number of operations whether the equation is solvable in rational integers*.

Word Equations

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Word equation: $P = Q$ where $P, Q \in (A_n \cup U)^*$

Solution: words $V_1, \ldots, V_m \in A_n^*$ such that

$$P_{V_1,\ldots,V_m}^{\upsilon_1,\ldots,\upsilon_m} \equiv Q_{V_1,\ldots,V_m}^{\upsilon_1,\ldots,\upsilon_m}$$

From Words to Numbers

(First Numbering: Matrices)

W.o.l.g $n = 2$, i.e., $A_2 = \{\alpha_1, \alpha_2\}$

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Lemma. Every 2×2 matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with natural number elements and the determinant equal to 1 can be represented in a unique way as the product of matrices $M_1 = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ and $M_2 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

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$$\text{"}\alpha_{i_1} \ldots \alpha_{i_k}\text{"} \quad \sim \quad M_{i_1} \times \cdots \times M_{i_k}$$

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$$\begin{array}{rcl} ``\alpha_{i_1} \ldots \alpha_{i_k}" & \sim & M_{i_1} \times \cdots \times M_{i_k} \\ v_1, \ldots, v_m & \sim & a_1, b_1, c_1, d_1, \ldots, a_m, b_m, c_m, d_m \end{array}$$

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$$\begin{array}{rcl} "\alpha_{i_1} \ldots \alpha_{i_k}" & \sim & M_{i_1} \times \cdots \times M_{i_k} \\ v_1, \ldots, v_m & \sim & a_1, b_1, c_1, d_1, \ldots, a_m, b_m, c_m, d_m \end{array}$$

$$P = Q \quad \sim \quad \begin{array}{l} P_{11}(\ldots, a_i, b_i, c_i, d_i, \ldots,) = Q_{11}(\ldots, a_i, b_i, c_i, d_i, \ldots,) \\ P_{12}(\ldots, a_i, b_i, c_i, d_i, \ldots,) = Q_{12}(\ldots, a_i, b_i, c_i, d_i, \ldots,) \\ P_{21}(\ldots, a_i, b_i, c_i, d_i, \ldots,) = Q_{21}(\ldots, a_i, b_i, c_i, d_i, \ldots,) \\ P_{22}(\ldots, a_i, b_i, c_i, d_i, \ldots,) = Q_{22}(\ldots, a_i, b_i, c_i, d_i, \ldots,) \end{array}$$

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$$\begin{array}{rcl} ''\alpha_{i_1} \ldots \alpha_{i_k}'' & \sim & M_{i_1} \times \cdots \times M_{i_k} \\ \upsilon_1, \ldots, \upsilon_m & \sim & a_1, b_1, c_1, d_1, \ldots, a_m, b_m, c_m, d_m \end{array}$$

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From Words to Numbers

(Second Numbering: Fibonacci weights)

Every word $X = "\beta_{i_m}\beta_{i_{m-1}} \ldots \beta_{i_1}"$ in the binary alphabet $B = \{\beta_0, \beta_1\} = \{0, 1\}$ can be viewed as the number

$$x = i_m u_m + i_{m-1} u_{m-1} + \cdots + i_1 u_1$$

written in positional system with weights of digits being the Fibonacci numbers $u_1 = 1$, $u_2 = 1$, $u_3 = 2$, $u_4 = 3$, $\ldots$ (rather than traditional 1, 2, 4, 8, 16, $\ldots$).

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"0" ∼0	"00" ∼0	"000" ∼0	"0000" ∼0	"00000" ∼0
"1" ∼1	"01" ∼1	"001" ∼1	"0001" ∼1	"00001" ∼1
	"10" ∼1	"010" ∼1	"0010" ∼1	"00010" ∼1
	"11" ∼2	"011" ∼2	"0011" ∼2	"00011" ∼2
		"100" ∼2	"0100" ∼2	"00100" ∼2
		"101" ∼3	"0101" ∼3	"00101" ∼3
		"110" ∼3	"0110" ∼3	"00110" ∼3
		"111" ∼4	"0111" ∼4	"00111" ∼4

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"1"$\sim$1	"10"$\sim$1	"100"$\sim$2	"1000"$\sim$3	"10000"$\sim$5
	"11"$\sim$2	"101"$\sim$3	"1001"$\sim$4	"10001"$\sim$6
		"110"$\sim$3	"1010"$\sim$4	"10010"$\sim$6
		"111"$\sim$4	"1011"$\sim$5	"10011"$\sim$7
			"1100"$\sim$5	"11100"$\sim$8
			"1101"$\sim$6	"11101"$\sim$9
			"1110"$\sim$6	"11110"$\sim$9
			"1111"$\sim$7	"11111"$\sim$10

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	"110" $\sim$3	"1010" $\sim$4	"10010" $\sim$6
		"1100" $\sim$5	"10100" $\sim$7
		"1110" $\sim$6	"10110" $\sim$8
			"11000" $\sim$8
			"11010" $\sim$9
			"11100" $\sim$10
			"11110" $\sim$11

From Words to Numbers
(Second Numbering: Zeckendorf's words)

"" ∼0	"1010" ∼4	"100000" ∼8	"101010" ∼12
"10" ∼1	"10000" ∼5	"100010" ∼9	"1000000" ∼13
"100" ∼2	"10010" ∼6	"100100" ∼10	"1000010" ∼14
"1000" ∼3	"10100" ∼7	"101000" ∼11	"1000100" ∼15

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$''''\sim 0$	$''1010''\sim 4$	$''100000''\sim 8$	$''101010''\sim 12$
$''10''\sim 1$	$''10000''\sim 5$	$''100010''\sim 9$	$''1000000''\sim 13$
$''100''\sim 2$	$''10010''\sim 6$	$''100100''\sim 10$	$''1000010''\sim 14$
$''1000''\sim 3$	$''10100''\sim 7$	$''101000''\sim 11$	$''1000100''\sim 15$

Zeckendorf's Theorem. Every positive integer x can be represented in the form

$$x = i_m u_m + i_{m-1} u_{m-1} + \cdots + i_1 u_1$$

with additional restrictions $i_m = 1$, $i_1 = 0$, $i_{k+1} i_k = 0$, and and in unique way.

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Zeckendorf's words: they do not begin with "0", do not end with "1", and do not contain "11".

From Words to Numbers
(Third Numbering: Infinite Alphabet)

Every word "$\alpha_{i_k}\alpha_{i_{k-1}}\ldots\alpha_{i_1}$" in the infinite alphabet $A_\infty = \{\alpha_1, \alpha_2, \ldots\}$ can be presented by the Zeckendorf word

$$"10^{i_k}10^{i_{k-1}}\ldots10^{i_1}".$$

We get, via Fibonacci numbers, a natural one-to-one correspondence between words in the infinite alphabet A_∞ and natural numbers.

""$\sim$0	"1010"$\sim$4	"100000"$\sim$8	"101010"$\sim$12
"10"$\sim$1	"10000"$\sim$5	"100010"$\sim$9	"1000000"$\sim$13
"100"$\sim$2	"10010"$\sim$6	"100100"$\sim$10	"1000010"$\sim$14
"1000"$\sim$3	"10100"$\sim$7	"101000"$\sim$11	"1000100"$\sim$15
""$\sim$0	"$\alpha_1\alpha_1$"$\sim$4	"α_5"$\sim$8	"$\alpha_1\alpha_1\alpha_1$"$\sim$12
"α_1"$\sim$1	"α_4"$\sim$5	"$\alpha_3\alpha_1$"$\sim$9	"α_6"$\sim$13
"α_2"$\sim$2	"$\alpha_2\alpha_1$"$\sim$6	"$\alpha_2\alpha_2$"$\sim$10	"$\alpha_4\alpha_1$"$\sim$14
"α_3"$\sim$3	"$\alpha_1\alpha_2$"$\sim$7	"$\alpha_1\alpha_3$"$\sim$11	"$\alpha_3\alpha_2$"$\sim$15

From Words to Numbers
(Concatenation)

$$X = "\beta_{i_m} \ldots \beta_{i_1}" \qquad Y = "\beta_{j_n} \ldots \beta_{j_1}"$$

$$x = i_m u_m + \cdots + i_1 u_1 \qquad y = j_n u_n + \cdots + j_1 u_1$$

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(Concatenation)

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$$Z = XY = \text{"}\beta_{i_m} \dots \beta_{i_1} \beta_{j_n} \dots \beta_{j_1}\text{"}$$

$$z = i_m u_{m+n} + \cdots + i_1 u_{1+n} + j_n u_n + \cdots + j_1 u_1$$

From Words to Numbers
(Concatenation)

$$u_{k+n} = u_k u_{n+1} + u_{k-1} u_n$$

$$\begin{aligned} X &= \text{"}\beta_{i_m} \dots \beta_{i_1}\text{"} & Y &= \text{"}\beta_{j_n} \dots \beta_{j_1}\text{"} \\ x &= i_m u_m + \cdots + i_1 u_1 & y &= j_n u_n + \cdots + j_1 u_1 \end{aligned}$$

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$$Z = XY = \text{"}\beta_{i_m} \ldots \beta_{i_1} \beta_{j_n} \ldots \beta_{j_1}\text{"}$$

$$\begin{aligned} z &= i_m u_{m+n} + \cdots + i_1 u_{1+n} + j_n u_n + \cdots + j_1 u_1 \\ &= i_m(u_m u_{n+1} + u_{m-1} u_n) + \cdots + i_1(u_1 u_{n+1} + u_0 u_n) + y \end{aligned}$$

From Words to Numbers
(Concatenation)

$$\boxed{u_{k+n} = u_k u_{n+1} + u_{k-1} u_n}$$

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$$x = i_m u_m + \cdots + i_1 u_1 \qquad y = j_n u_n + \cdots + j_1 u_1$$

$$Z = XY = \text{"}\beta_{i_m} \ldots \beta_{i_1} \beta_{j_n} \ldots \beta_{j_1}\text{"}$$

$$\begin{aligned} z &= i_m u_{m+n} + \cdots + i_1 u_{1+n} + j_n u_n + \cdots + j_1 u_1 \\ &= i_m (u_m u_{n+1} + u_{m-1} u_n) + \cdots + i_1 (u_1 u_{n+1} + u_0 u_n) + y \\ &= (i_m u_m + \cdots + i_1 u_1) u_{n+1} + (i_m u_{m-1} + \cdots + i_1 u_0) u_n + y \end{aligned}$$

From Words to Numbers
(Concatenation)

$$u_{k+n} = u_k u_{n+1} + u_{k-1} u_n$$

$$X = "\beta_{i_m} \ldots \beta_{i_1}" \qquad Y = "\beta_{j_n} \ldots \beta_{j_1}"$$

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$$\begin{aligned} z &= i_m u_{m+n} + \cdots + i_1 u_{1+n} + j_n u_n + \cdots + j_1 u_1 \\ &= i_m(u_m u_{n+1} + u_{m-1} u_n) + \cdots + i_1(u_1 u_{n+1} + u_0 u_n) + y \\ &= \underbrace{(i_m u_m + \cdots + i_1 u_1)}_{x} u_{n+1} + (i_m u_{m-1} + \cdots + i_1 u_0) u_n + y \end{aligned}$$

From Words to Numbers
(Concatenation)

$$\boxed{u_{k+n} = u_k u_{n+1} + u_{k-1} u_n}$$

$$\begin{aligned} X &= \text{"}\beta_{i_m} \ldots \beta_{i_1}\text{"} \\ x &= i_m u_m + \cdots + i_1 u_1 \end{aligned} \qquad \begin{aligned} Y &= \text{"}\beta_{j_n} \ldots \beta_{j_1}\text{"} \\ y &= j_n u_n + \cdots + j_1 u_1 \end{aligned}$$

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From Words to Numbers (Concatenation)

$$\boxed{u_{k+n} = u_k u_{n+1} + u_{k-1} u_n}$$

$$\begin{aligned} X &= \text{"}\beta_{i_m} \dots \beta_{i_1}\text{"} & Y &= \text{"}\beta_{j_n} \dots \beta_{j_1}\text{"} \\ x &= i_m u_m + \dots + i_1 u_1 & y &= j_n u_n + \dots + j_1 u_1 \end{aligned}$$

$$Z = XY = \text{"}\beta_{i_m} \dots \beta_{i_1} \beta_{j_n} \dots \beta_{j_1}\text{"}$$

$$\begin{aligned} z &= i_m u_{m+n} + \dots + i_1 u_{1+n} + j_n u_n + \dots + j_1 u_1 \\ &= i_m(u_m u_{n+1} + u_{m-1} u_n) + \dots + i_1(u_1 u_{n+1} + u_0 u_n) + y \\ &= \underbrace{(i_m u_m + \dots + i_1 u_1)}_{x} u_{n+1} + \underbrace{(i_m u_{m-1} + \dots + i_1 u_0)}_{x_1} u_n + y \\ &= x y_3 + x_1 y_2 + y \end{aligned}$$

where

$$y_2 = u_n, \quad y_3 = u_{n+1}$$

From Words to Numbers
(Concatenation cont.)

$$
\begin{aligned}
X &= ''\beta_{i_m} \ldots \beta_{i_1}'' \\
x &= i_m u_m + \cdots + i_1 u_1 \\
x_1 &= i_m u_{m-1} + \cdots + i_1 u_0 \\
x_2 &= u_m \\
x_3 &= u_{m+1}
\end{aligned}
\qquad
\begin{aligned}
Y &= ''\beta_{j_n} \ldots \beta_{j_1}'' \\
y &= j_n u_n + \cdots + j_1 u_1 \\
y_1 &= j_n u_{n-1} + \cdots + j_1 u_0 \\
y_2 &= u_n \\
y_3 &= u_{n+1}
\end{aligned}
$$

From Words to Numbers
(Concatenation cont.)

$$\begin{aligned} X &= \text{"}\beta_{i_m} \dots \beta_{i_1}\text{"} \\ x &= i_m u_m + \dots + i_1 u_1 \\ x_1 &= i_m u_{m-1} + \dots + i_1 u_0 \\ x_2 &= u_m \\ x_3 &= u_{m+1} \end{aligned} \qquad \begin{aligned} Y &= \text{"}\beta_{j_n} \dots \beta_{j_1}\text{"} \\ y &= j_n u_n + \dots + j_1 u_1 \\ y_1 &= j_n u_{n-1} + \dots + j_1 u_0 \\ y_2 &= u_n \\ y_3 &= u_{n+1} \end{aligned}$$

$$Z = XY \quad = \quad \text{"}\beta_{i_m} \dots \beta_{i_1}\beta_{j_n} \dots \beta_{j_1}\text{"}$$

From Words to Numbers
(Concatenation cont.)

$$\begin{aligned} X &= \text{"}\beta_{i_m} \ldots \beta_{i_1}\text{"} \\ x &= i_m u_m + \cdots + i_1 u_1 \\ x_1 &= i_m u_{m-1} + \cdots + i_1 u_0 \\ x_2 &= u_m \\ x_3 &= u_{m+1} \end{aligned}$$

$$\begin{aligned} Y &= \text{"}\beta_{j_n} \ldots \beta_{j_1}\text{"} \\ y &= j_n u_n + \cdots + j_1 u_1 \\ y_1 &= j_n u_{n-1} + \cdots + j_1 u_0 \\ y_2 &= u_n \\ y_3 &= u_{n+1} \end{aligned}$$

$$\begin{aligned} Z = XY &= \text{"}\beta_{i_m} \ldots \beta_{i_1} \beta_{j_n} \ldots \beta_{j_1}\text{"} \\ z &= x_1 y_2 + x y_3 + y \end{aligned}$$

From Words to Numbers
(Concatenation cont.)

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\qquad
\begin{aligned}
Y &= \text{"}\beta_{j_n} \dots \beta_{j_1}\text{"} \\
y &= j_n u_n + \dots + j_1 u_1 \\
y_1 &= j_n u_{n-1} + \dots + j_1 u_0 \\
y_2 &= u_n \\
y_3 &= u_{n+1}
\end{aligned}
$$

$$
\begin{aligned}
Z = XY &= \text{"}\beta_{i_m} \dots \beta_{i_1}\beta_{j_n} \dots \beta_{j_1}\text{"} \\
z &= x_1 y_2 + x y_3 + y \\
z_1 &= x_1(y_3 - y_2) + x y_2 + y_1
\end{aligned}
$$

From Words to Numbers
(Concatenation cont.)

$$\begin{aligned} X &= ``\beta_{i_m} \ldots \beta_{i_1}" \\ x &= i_m u_m + \cdots + i_1 u_1 \\ x_1 &= i_m u_{m-1} + \cdots + i_1 u_0 \\ x_2 &= u_m \\ x_3 &= u_{m+1} \end{aligned} \qquad \begin{aligned} Y &= ``\beta_{j_n} \ldots \beta_{j_1}" \\ y &= j_n u_n + \cdots + j_1 u_1 \\ y_1 &= j_n u_{n-1} + \cdots + j_1 u_0 \\ y_2 &= u_n \\ y_3 &= u_{n+1} \end{aligned}$$

$$\begin{aligned} Z = XY &= ``\beta_{i_m} \ldots \beta_{i_1} \beta_{j_n} \ldots \beta_{j_1}" \\ z &= x_1 y_2 + x y_3 + y \\ z_1 &= x_1 (y_3 - y_2) + x y_2 + y_1 \\ z_2 &= (x_3 - x_2) y_2 + x_2 y_3 \end{aligned}$$

From Words to Numbers
(Concatenation cont.)

$$\begin{aligned} X &= ``\beta_{i_m} \dots \beta_{i_1}" \\ x &= i_m u_m + \cdots + i_1 u_1 \\ x_1 &= i_m u_{m-1} + \cdots + i_1 u_0 \\ x_2 &= u_m \\ x_3 &= u_{m+1} \end{aligned}$$

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$$\begin{aligned} Z = XY &= ``\beta_{i_m} \dots \beta_{i_1} \beta_{j_n} \dots \beta_{j_1}" \\ z &= x_1 y_2 + x y_3 + y \\ z_1 &= x_1 (y_3 - y_2) + x y_2 + y_1 \\ z_2 &= (x_3 - x_2) y_2 + x_2 y_3 \\ z_3 &= x_2 y_2 + x_3 y_3 \end{aligned}$$

Fibonacci Numbers and Diophantine Equations

Theorem (G. D. Cassini [1680]).

$$u_{m+1}^2 - u_{m+1}u_m - u_m^2 = (-1)^m$$

Fibonacci Numbers and Diophantine Equations

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$$u_{m+1}^2 - u_{m+1}u_m - u_m^2 = (-1)^m$$

Theorem (M. J. Wasteels [1902]). If

$$w^2 - wv - v^2 = \pm 1$$

then

$$w = u_{m+1}, \quad v = u_m$$

for some m.

From Words to Numbers
(Second Numbering cont.)

$$\begin{aligned} X &= \text{"}\beta_{i_m} \ldots \beta_{i_1}\text{"} \\ x &= i_m u_m + \cdots + i_1 u_1 \\ x_1 &= i_m u_{m-1} + \cdots + i_1 u_0 \\ x_2 &= u_m \\ x_3 &= u_{m+1} \end{aligned}$$

From Words to Numbers
(Second Numbering cont.)

$$
\begin{aligned}
X &= \text{"}\beta_{i_m} \ldots \beta_{i_1}\text{"} \\
x &= i_m u_m + \cdots + i_1 u_1 \\
x_1 &= i_m u_{m-1} + \cdots + i_1 u_0 \\
x_2 &= u_m \\
x_3 &= u_{m+1}
\end{aligned}
$$

$$(x_3^2 - x_3 x_2 - x_2^2)^2 = 1$$

From Words to Numbers
(Second Numbering cont.)

$$\begin{aligned} X &= \text{"}\beta_{i_m} \ldots \beta_{i_1}\text{"} \\ x &= i_m u_m + \cdots + i_1 u_1 \\ x_1 &= i_m u_{m-1} + \cdots + i_1 u_0 \\ x_2 &= u_m \\ x_3 &= u_{m+1} \end{aligned}$$

$$(x_3^2 - x_3 x_2 - x_2^2)^2 = 1$$
$$x_2 \leq x < x_3$$

From Words to Numbers

(Second Numbering cont.)

$$\frac{u_k}{u_{k-1}} \approx \phi = \frac{1+\sqrt{5}}{2}$$

$$\begin{aligned} X &= \text{"}\beta_{i_m} \ldots \beta_{i_1}\text{"} \\ x &= i_m u_m + \cdots + i_1 u_1 \\ x_1 &= i_m u_{m-1} + \cdots + i_1 u_0 \\ x_2 &= u_m \\ x_3 &= u_{m+1} \end{aligned}$$

$$(x_3^2 - x_3 x_2 - x_2^2)^2 = 1$$

$$x_2 \leq x < x_3$$

From Words to Numbers
(Second Numbering cont.)

$$\frac{u_k}{u_{k-1}} \approx \phi = \frac{1+\sqrt{5}}{2}$$

$$\begin{aligned} X &= \text{"}\beta_{i_m} \ldots \beta_{i_1}\text{"} \\ x &= i_m u_m + \cdots + i_1 u_1 \\ x_1 &= i_m u_{m-1} + \cdots + i_1 u_0 \\ x_2 &= u_m \\ x_3 &= u_{m+1} \end{aligned}$$

$$(x_3^2 - x_3 x_2 - x_2^2)^2 = 1$$
$$x_2 \le x < x_3$$

$\frac{x}{x_1} \approx \phi$

From Words to Numbers
(Second Numbering cont.)

$$\frac{u_k}{u_{k-1}} \approx \phi = \frac{1+\sqrt{5}}{2}$$

$$\begin{aligned}
X &= \text{"}\beta_{i_m} \ldots \beta_{i_1}\text{"} \\
x &= i_m u_m + \cdots + i_1 u_1 \\
x_1 &= i_m u_{m-1} + \cdots + i_1 u_0 \\
x_2 &= u_m \\
x_3 &= u_{m+1}
\end{aligned}$$

$$(x_3^2 - x_3 x_2 - x_2^2)^2 = 1$$

$$x_2 \leq x < x_3$$

$\frac{x}{x_1} \approx \phi$

$$\phi - 2 < x_1 - x/\phi < \phi - 1$$

Concatenation

$$\boxed{\mathrm{GCD}(u_m, u_n) = u_{\mathrm{GCD}(m,n)}}$$

$$\begin{array}{rcl} X & = & "\beta_{i_m} \ldots \beta_{i_1}" \\ x & = & i_m u_m + \cdots + i_1 u_1 \\ x_1 & = & i_m u_{m-1} + \cdots + i_1 u_0 \\ x_2 & = & u_m \\ x_3 & = & u_{m+1} \end{array} \qquad \begin{array}{rcl} Y & = & "\beta_{j_n} \ldots \beta_{j_1}" \\ y & = & j_n u_n + \cdots + j_1 u_1 \\ y_1 & = & j_n u_{n-1} + \cdots + j_1 u_0 \\ y_2 & = & u_n \\ y_3 & = & u_{n+1} \end{array}$$

$$\begin{array}{rcl} \mathrm{length}(X) = \mathrm{length}(Y) & \Leftrightarrow & x_2 = y_2 \\ \mathrm{length}(X) \big| \mathrm{length}(Y) & \Leftrightarrow & x_2(x_2 + x_3) \big| y_2(y_2 + y_3) \\ \mathrm{GCD}(\mathrm{length}(X), \mathrm{length}(Y)) = 1 & \Leftrightarrow & \mathrm{GCD}(x_2(x_2 + x_3), y_2(y_2 + y_3)) = \end{array}$$

$$"10^{i_k}10^{i_{k-1}}\dots 10^{i_1}"$$

$$\mathrm{length}("10^{i_k}10^{i_{k-1}}\dots 10^{i_1}") = k + i_k + i_{k-1} + \cdots + i_1$$

Concatenation

$$\frac{u_k}{u_{k-1}} \approx \phi = \frac{1+\sqrt{5}}{2}$$

$$\begin{aligned}
X &= \text{"}\beta_{i_m}\beta_{i_{m-1}}\ldots\beta_{i_1}\text{"} \qquad \text{"1"} \mapsto \text{"10"}, \ \text{"0"} \mapsto \text{"} \\
& \quad \text{"}\beta_{i_m}0\beta_{i_{m-1}}0\ldots\beta_{i_1}0\text{"} \\
x &= i_m u_{2m} + i_{m-1}u_{2(m-1)} + \cdots + i_1 u_2 \\
x_1 &= i_m u_{2m-1} + i_{m-1}u_{2(m-1)-1} + \cdots + i_1 u_1 \\
x_2 &= u_{2m} \\
x_3 &= u_{2m+1}
\end{aligned}$$

$$x_3^2 - x_3x_2 - x_2^2 = 1$$

$$x < x_3$$

$$\phi - 2 < x_1 - x/\phi < \phi - 1$$

$$\text{"10"} \mapsto \text{"1"}, \ \text{"00"} \mapsto \text{"0"}, \ \text{"01"} \mapsto \text{"1"}$$