# Eigenvalues and eigenvectors of some Hankel matrices related to the zeta and $L$-functions

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http://logic.pdmi.ras.ru/~yumat/personaljournal/zetahiddenlife/zetahiddenlife.html

# Classical Statement of Riemann's Hypothesis

The Riemann's zeta function:

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

**Riemann's Hypothesis (version 1).** *All non-trivial zeros of function* $\zeta(z)$ *lie on the critical line* $\mathrm{Re}(z) = \frac{1}{2}$.

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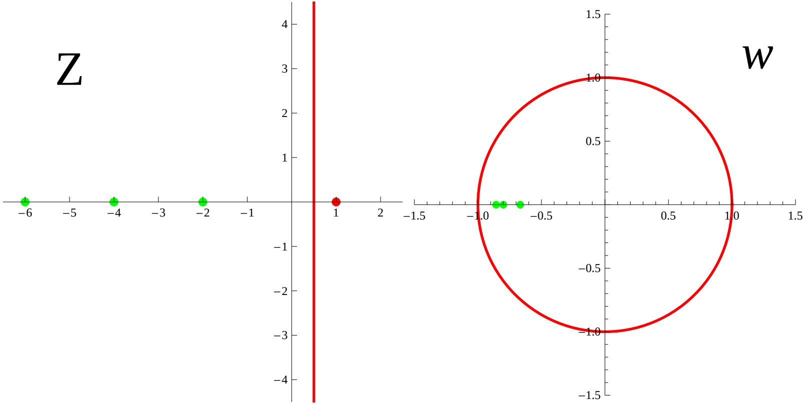

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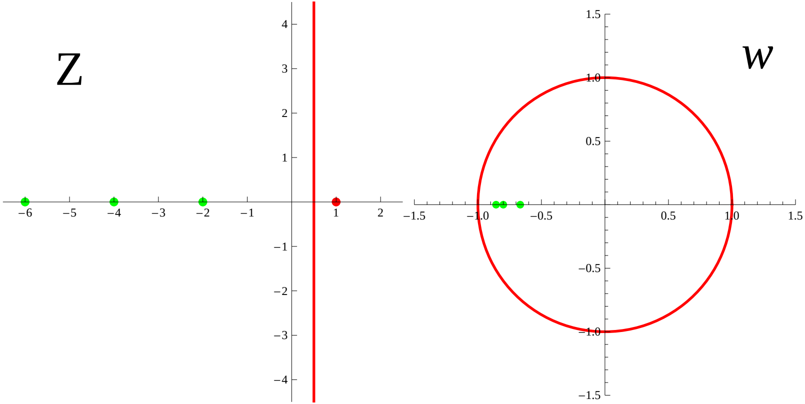


**RH (version 3)** *The trivial zeros*

$$\mathrm{w}_1 = -\frac{2}{3}, \mathrm{w}_2 = -\frac{4}{5}, \ldots, \mathrm{w}_n = -\frac{2n}{2n+1}, \ldots$$

*are the only zeros of function* $\tilde{\zeta}(w) = \zeta^*(\frac{w}{w+1})$ *lying in the open disk* $|w| < 1$.

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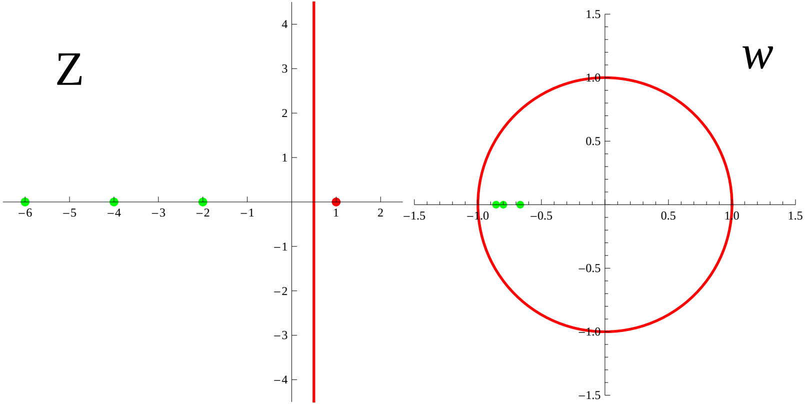

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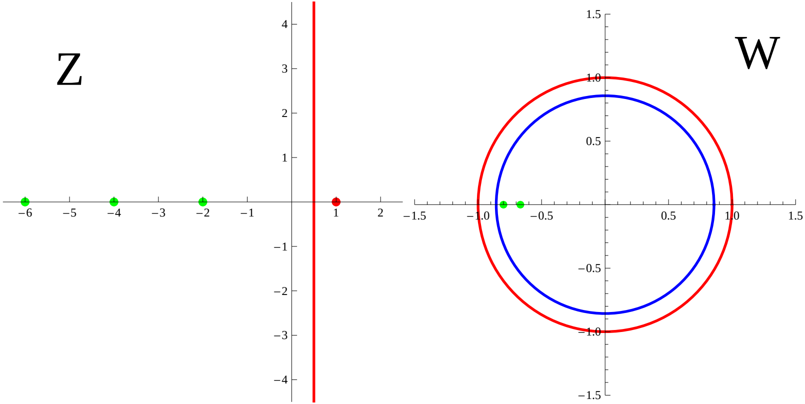


**RH (version 4).** *For every n the trivial zeros*

$$w_1 = -\frac{2}{3}, w_2 = -\frac{4}{5}, \ldots, w_n = -\frac{2n}{2n+1}$$

*are the only zeros of function* $\tilde{\zeta}(w)$ *lying in the closed disk* $|w| \le \frac{2n+1}{2n+2}$.

# Padé approximations

$$\tilde{\zeta}(w) \approx \frac{P_{n,m}(w)}{Q_{n,m}(w)} = \frac{1 + p_{n,m,1}w + \cdots + p_{n,m,n}w^n}{1 + q_{n,m,1}w + \cdots + q_{n,m,m}w^m}$$

$$= \tilde{\zeta}(w) + O(w^{n+m+1})$$

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**Theorem (de Montessus de Ballore, [1905]).** *For $m \to \infty$*

$$P_{n,m}(w) \to \prod_{j=1}^{n} \left(1 - \frac{w}{v_j}\right)$$

*where $v_1, \ldots, v_n$ are zeroes of $\tilde{\zeta}$ with least absolute values (provided that the choice of such zeroes is unique).*

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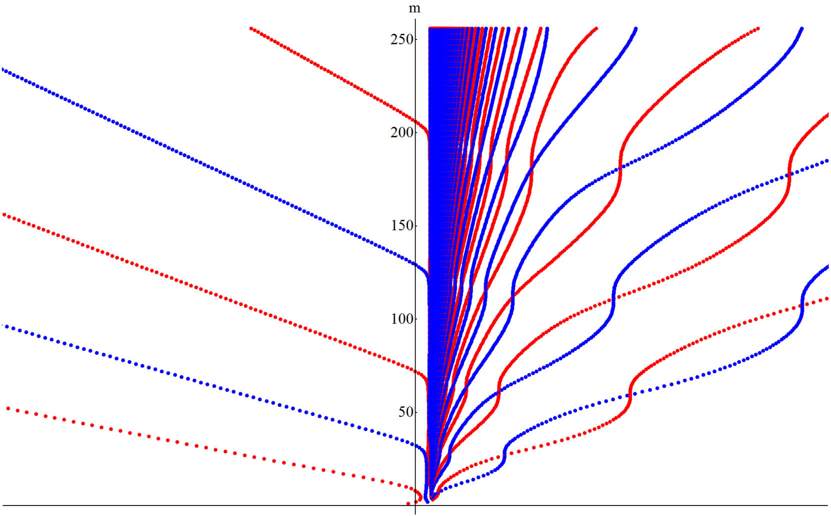
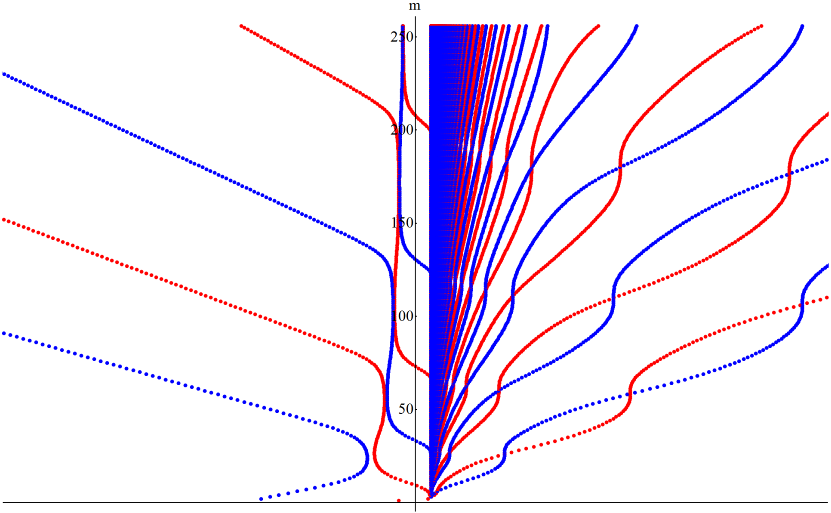



Values of $\log|\mu_{1,m,k}|$, red for $\mu_{1,m,k} > 0$ and blue for $\mu_{1,m,k} < 0$ for $m = 1 \ldots 256$

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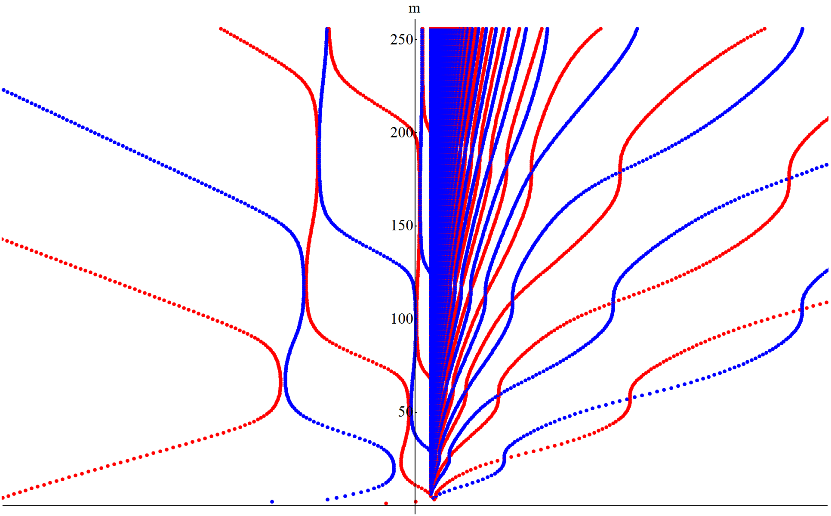


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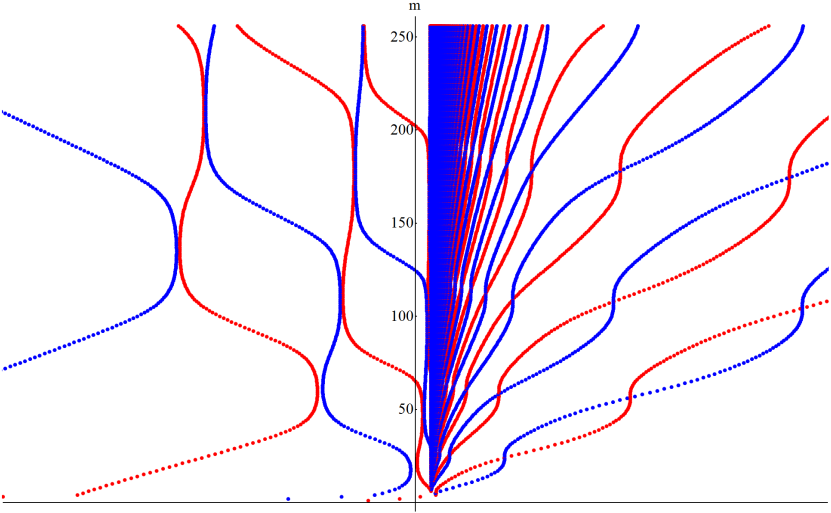
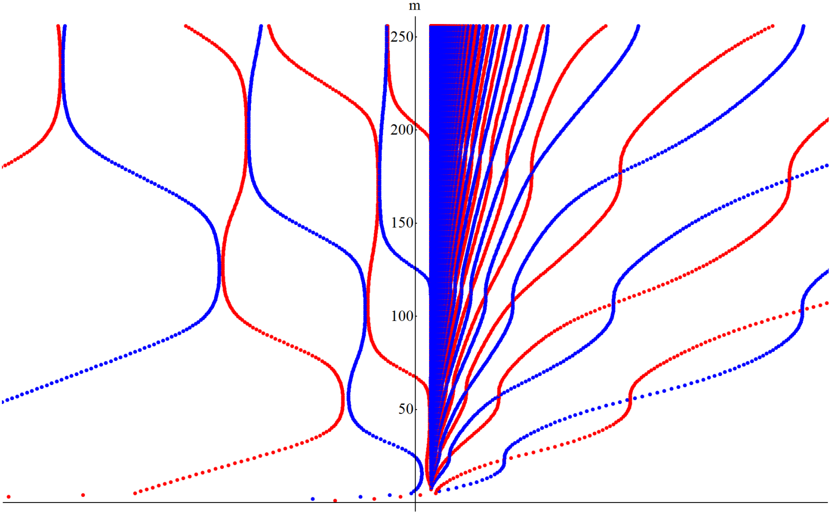



Values of $\log|\mu_{4,m,k}|$, red for $\mu_{4,m,k} > 0$ and blue for $\mu_{4,m,k} < 0$ for $m = 1 \ldots 256$

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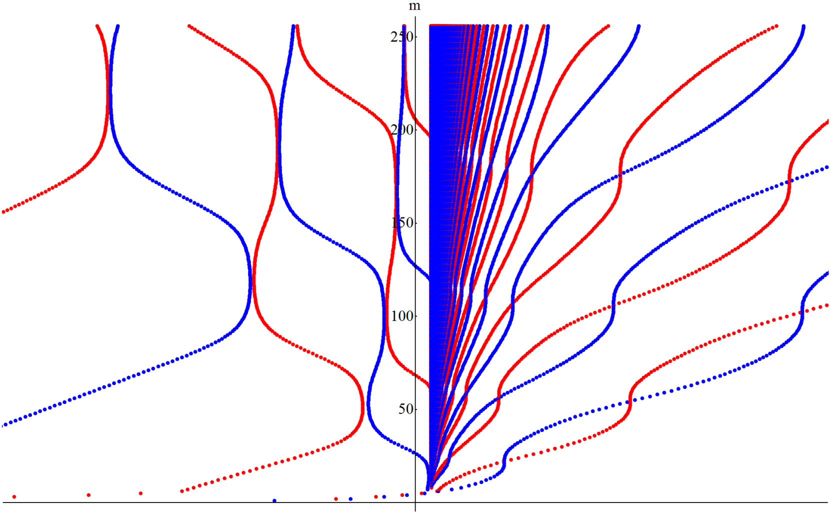


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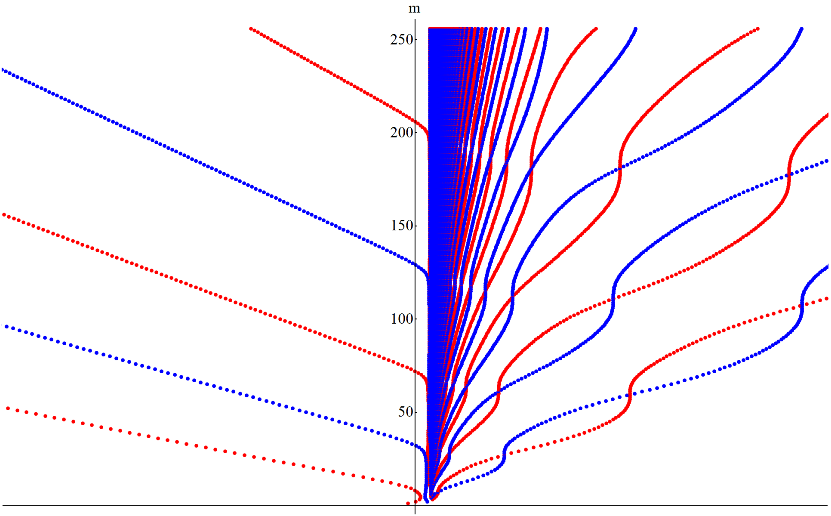


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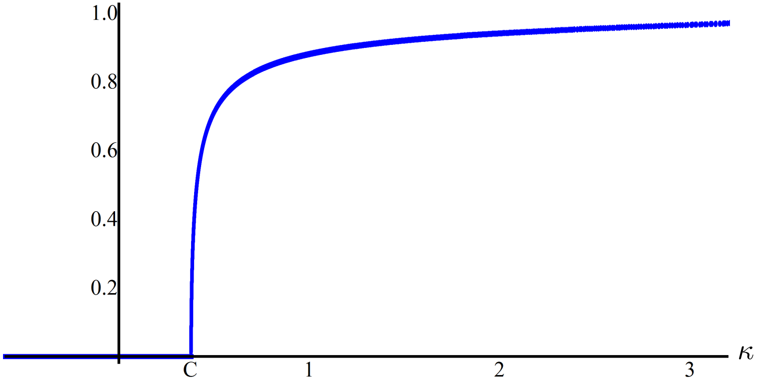
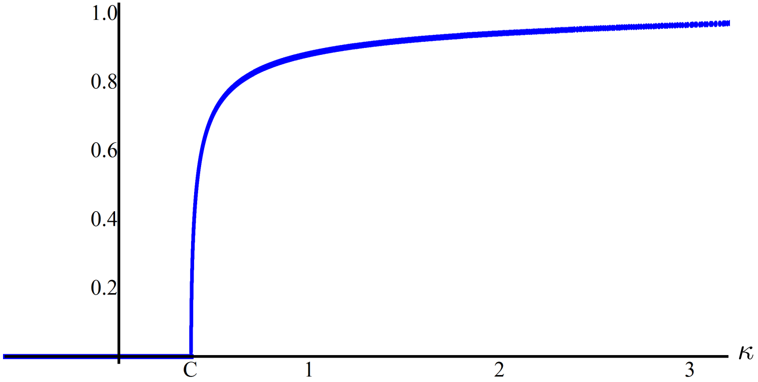
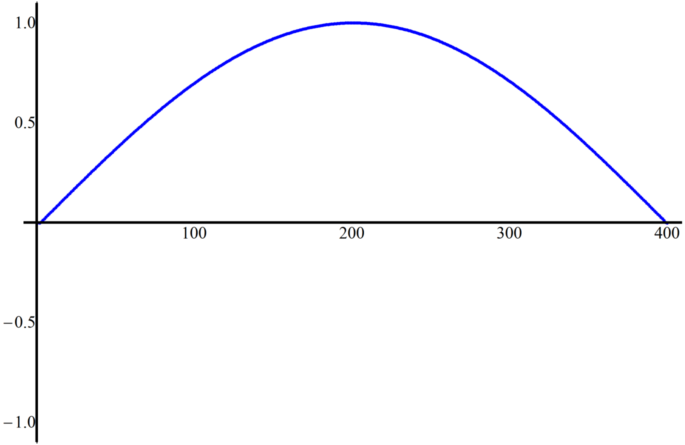



Limiting percetage of eigenvalues $\mu_{n,m,k}$ such that $\ln|\mu_{n,m,k}| < \kappa$



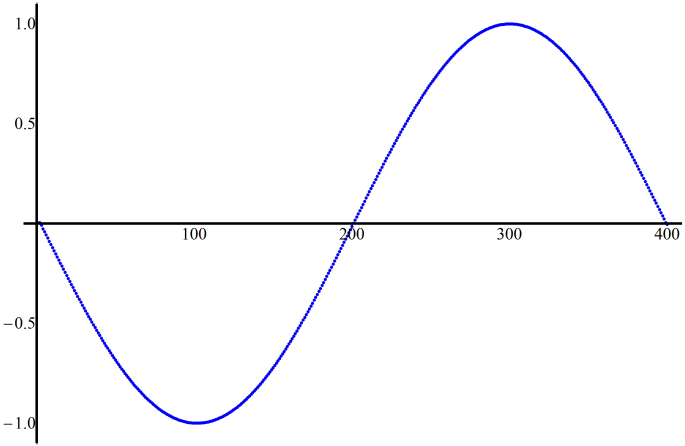

Limiting percetage of eigenvalues $\mu_{n,m,k}$ such that $\ln|\mu_{n,m,k}| < \kappa$

$$C = 0.378679220... = \ln\left(-\zeta\left(\frac{1}{2}\right)\right)$$



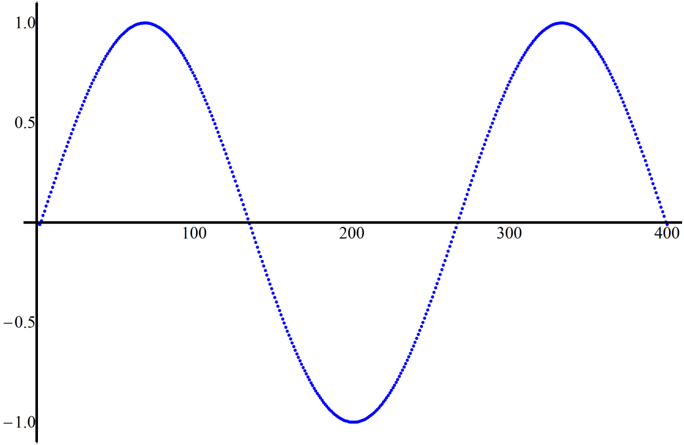

Entries of the normalized eigenvector for the eigenvalue

$$\mu_{1,400,9} = -1.4603552\ldots$$
$$\zeta(.5) = -1.4603545\ldots$$



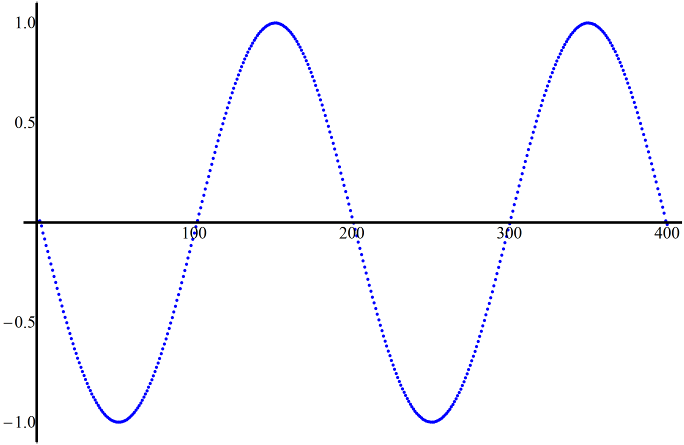

Entries of the normalized eigenvector for the eigenvalue

$$\begin{aligned}\mu_{1,400,10} &= 1.4603574\ldots \\ -\zeta(.5) &= 1.4603545\ldots\end{aligned}$$



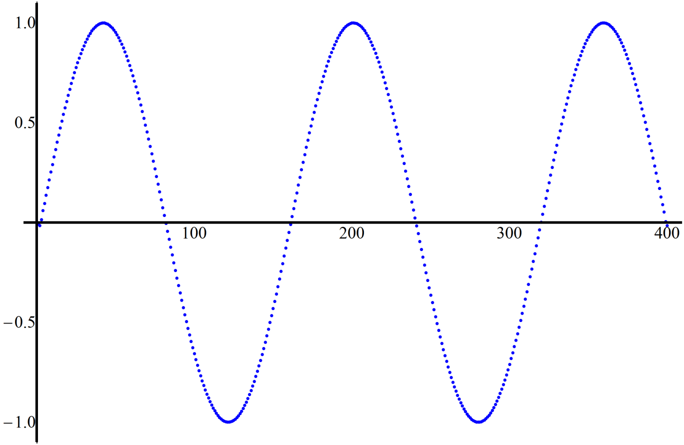

Entries of the normalized eigenvector for the eigenvalue

$$\mu_{1,400,11} = -1.4603610\ldots$$
$$\zeta(.5) = -1.4603545\ldots$$



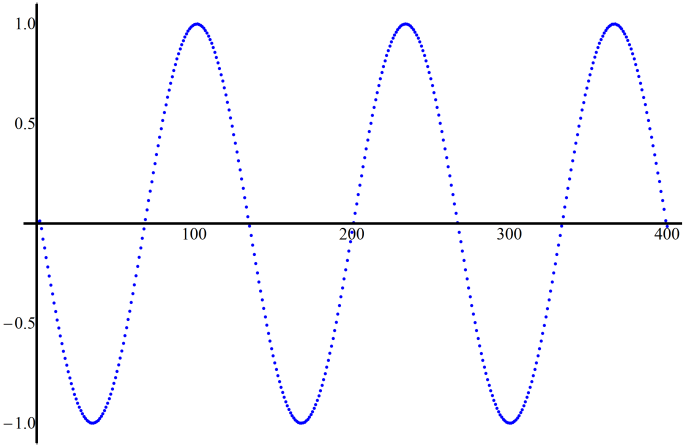

Entries of the normalized eigenvector for the eigenvalue

$$\mu_{1,400,12} = 1.4603660\ldots$$
$$-\zeta(.5) = 1.4603545\ldots$$



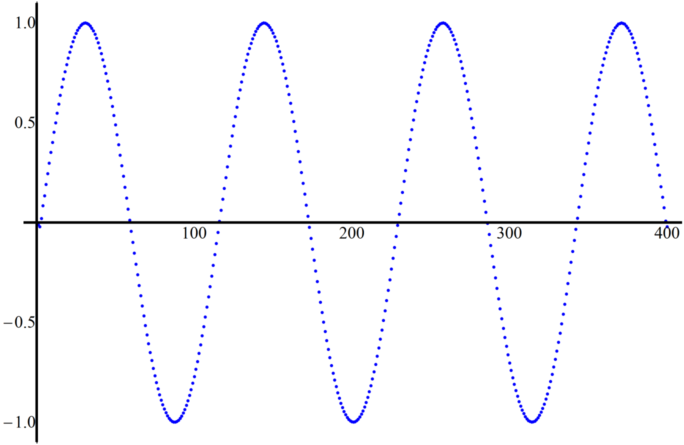

Entries of the normalized eigenvector for the eigenvalue

$$\mu_{1,400,13} = -1.4603726\ldots$$
$$\zeta(.5) = -1.4603545\ldots$$



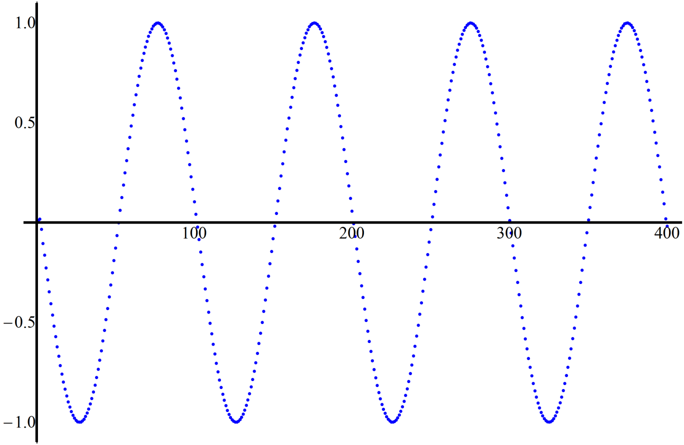

Entries of the normalized eigenvector for the eigenvalue

$$\mu_{1,400,14} = 1.4603805...$$
$$-\zeta(.5) = 1.4603545...$$



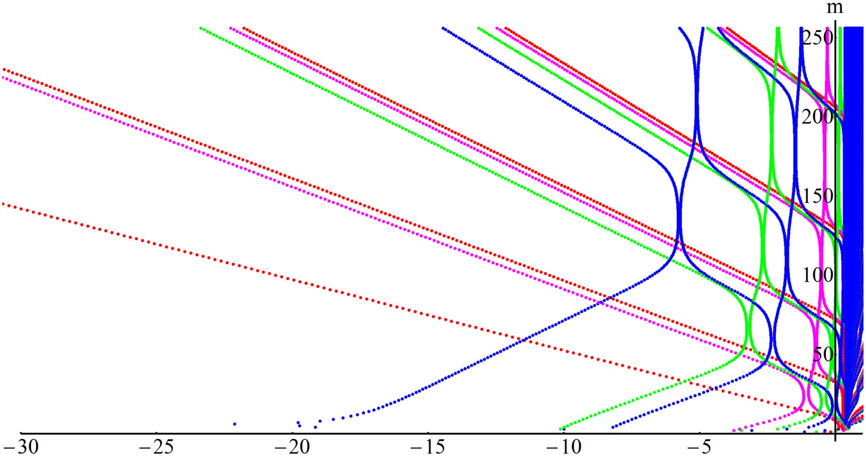

Entries of the normalized eigenvector for the eigenvalue

$$\mu_{1,400,15} = -1.4603900\ldots$$
$$\zeta(.5) = -1.4603545\ldots$$




Entries of the normalized eigenvector for the eigenvalue

$$\mu_{1,400,16} = 1.4604008...$$
$$-\zeta(.5) = 1.4603545...$$




Values of $\log|\mu_{n,m,k}|$ for $n = 1, 2, 3, 4$ for $m = 1, \ldots, 256$

**Conjecture A.** *For every $n$, $k$ there is a rational number $A_{n,k}$ such that for $m \to \infty$*

$$|\mu_{n,m,k}| = \left| (A_{n,k} + o(1)) \, \frac{\mathrm{d}}{\mathrm{d}w} \tilde{\zeta}(w) \Big|_{w = \mathrm{w}_{n+k}} \mathrm{w}_{n+k}^{-m} \right|$$

$w_n = -\frac{2n}{2n+1}$ is a trivial zero of $\zeta\left(\frac{w}{w+1}\right)$

**Conjecture A.** *For every $n$, $k$ there is a rational number $A_{n,k}$ such that for $m \to \infty$*

$$|\mu_{n,m,k}| = \left| (A_{n,k} + o(1)) \left. \frac{\mathrm{d}}{\mathrm{d}w}\tilde{\zeta}(w) \right|_{w=\mathrm{w}_{n+k}} \mathrm{w}_{n+k}^{-m} \right|$$

$w_n = -\frac{2n}{2n+1}$ is a trivial zero of $\zeta\left(\frac{w}{w+1}\right)$

**Numerical data.** For $n = 0,\ m = 300,\ k = 1$

$$\left| \frac{\mu_{n,m,k}}{\left. \frac{\mathrm{d}}{\mathrm{d}w}\tilde{\zeta}(w) \right|_{w=\mathrm{w}_{n+k}} \mathrm{w}_{n+k}^{-m}} \right| = 0.55555555555555555555555566973919400\ldots$$

**Conjecture A.** *For every $n$, $k$ there is a rational number $A_{n,k}$ such that for $m \to \infty$*

$$|\mu_{n,m,k}| = \left| (A_{n,k} + o(1)) \left. \frac{\mathrm{d}}{\mathrm{d}w} \tilde{\zeta}(w) \right|_{w = \mathrm{w}_{n+k}} \mathrm{w}_{n+k}^{-m} \right|$$

$w_n = -\frac{2n}{2n+1}$ is a trivial zero of $\zeta\left(\frac{w}{w+1}\right)$

**Numerical data.** For $n = 0,\ m = 300,\ k = 1$

$$\left| \frac{\mu_{n,m,k}}{\left. \frac{\mathrm{d}}{\mathrm{d}w} \tilde{\zeta}(w) \right|_{w = \mathrm{w}_{n+k}} \mathrm{w}_{n+k}^{-m}} \right| = 0.5555555555555555555555555669739194 00\ldots$$

**Numerical Conjecture A.**

$$A_{0,1} = \frac{5}{9}$$

**Conjecture A.** *For every $n$, $k$ there is a rational number $A_{n,k}$ such that for $m \to \infty$*

$$|\mu_{n,m,k}| = \left| (A_{n,k} + o(1)) \frac{\mathrm{d}}{\mathrm{d}w} \tilde{\zeta}(w) \Big|_{w=\mathrm{w}_{n+k}} \mathrm{w}_{n+k}^{-m} \right|$$

**Numerical data.** For $n = 0,\ m = 512,\ k = 2$

$$\left| \frac{\mu_{n,m,k}}{\frac{\mathrm{d}}{\mathrm{d}w} \tilde{\zeta}(w) \Big|_{w=\mathrm{w}_{n+k}} \mathrm{w}_{n+k}^{-m}} \right| = 4.41000000000000314765317394722...$$

**Conjecture A.** *For every n, k there is a rational number $A_{n,k}$ such that for $m \to \infty$*

$$|\mu_{n,m,k}| = \left| (A_{n,k} + o(1)) \, \frac{\mathrm{d}}{\mathrm{d}w}\tilde{\zeta}(w)\Big|_{w=\mathrm{w}_{n+k}} \mathrm{w}_{n+k}^{-m} \right|$$

**Numerical data.** For $n = 0,\ m = 512,\ k = 2$

$$\left| \frac{\mu_{n,m,k}}{\frac{\mathrm{d}}{\mathrm{d}w}\tilde{\zeta}(w)\Big|_{w=\mathrm{w}_{n+k}} \mathrm{w}_{n+k}^{-m}} \right| = 4.41000000000000314765317394722\ldots$$

**Numerical Conjecture A.**

$$A_{0,2} = \frac{441}{100} = \frac{3^2 \times 7^2}{2^2 \times 5^2}$$

**Conjecture A.** *For every $n$, $k$ there is a rational number $A_{n,k}$ such that for $m \to \infty$*

$$|\mu_{n,m,k}| = \left| (A_{n,k} + o(1)) \left. \frac{\mathrm{d}}{\mathrm{d}w} \tilde{\zeta}(w) \right|_{w=\mathrm{w}_{n+k}} \mathrm{w}_{n+k}^{-m} \right|$$

**Numerical data.** For $n = 0,\ m = 1000,\ k = 3$

$$\left| \frac{\mu_{n,m,k}}{\left. \frac{\mathrm{d}}{\mathrm{d}w} \tilde{\zeta}(w) \right|_{w=\mathrm{w}_{n+k}} \mathrm{w}_{n+k}^{-m}} \right| \quad = \quad 40.62914540816327697105916177659280 9\ldots$$

**Conjecture A.** *For every $n$, $k$ there is a rational number $A_{n,k}$ such that for $m \to \infty$*

$$|\mu_{n,m,k}| = \left| (A_{n,k} + o(1)) \left. \frac{\mathrm{d}}{\mathrm{d}w} \tilde{\zeta}(w) \right|_{w=\mathrm{w}_{n+k}} \mathrm{w}_{n+k}^{-m} \right|$$

**Numerical data.** For $n = 0,\ m = 1000,\ k = 3$

$$\left| \frac{\mu_{n,m,k}}{\left. \frac{\mathrm{d}}{\mathrm{d}w} \tilde{\zeta}(w) \right|_{w=\mathrm{w}_{n+k}} \mathrm{w}_{n+k}^{-m}} \right| = 40.629145408163276971059161776592809\ldots$$

$$= [40, 1, 1, 1, 2, 3, 2, 1, 1, 6, 3, 8716968, 1, 3, 15, \ldots]$$

**Conjecture A.** *For every n, k there is a rational number $A_{n,k}$ such that for $m \to \infty$*

$$|\mu_{n,m,k}| = \left| (A_{n,k} + o(1)) \, \frac{\mathrm{d}}{\mathrm{d}w}\tilde{\zeta}(w)\Big|_{w=\mathrm{w}_{n+k}} \mathrm{w}_{n+k}^{-m} \right|$$

**Numerical data.** For $n = 0, \ m = 1000, \ k = 3$

$$\left| \frac{\mu_{n,m,k}}{\frac{\mathrm{d}}{\mathrm{d}w}\tilde{\zeta}(w)\Big|_{w=\mathrm{w}_{n+k}} \mathrm{w}_{n+k}^{-m}} \right| = 40.629145408163276971059161776592809...$$

$$= [40, 1, 1, 1, 2, 3, 2, 1, 1, 6, 3, 8716968, 1, 3, 15, ...]$$

For $n = 0, \ m = 1024, \ k = 3$

$$\left| \frac{\mu_{n,m,k}}{\frac{\mathrm{d}}{\mathrm{d}w}\tilde{\zeta}(w)\Big|_{w=\mathrm{w}_{n+k}} \mathrm{w}_{n+k}^{-m}} \right| = 40.629145408163270179386846358800...$$

$$= [40, 1, 1, 1, 2, 3, 2, 1, 1, 6, 3, 20865457, 2, 6, 1, ...]$$

**Conjecture A.** *For every $n$, $k$ there is a rational number $A_{n,k}$ such that for $m \to \infty$*

$$|\mu_{n,m,k}| = \left| (A_{n,k} + o(1)) \, \frac{\mathrm{d}}{\mathrm{d}w}\tilde{\zeta}(w)\bigg|_{w=\mathrm{w}_{n+k}} \mathrm{w}_{n+k}^{-m} \right|$$

**Numerical data.** For $n = 0,\ m = 1000,\ k = 3$

$$\left| \frac{\mu_{n,m,k}}{\frac{\mathrm{d}}{\mathrm{d}w}\tilde{\zeta}(w)\Big|_{w=\mathrm{w}_{n+k}} \mathrm{w}_{n+k}^{-m}} \right| = 40.629145408163276971059161776592809...$$

$$= [40, 1, 1, 1, 2, 3, 2, 1, 1, 6, 3, 8716968, 1, 3, 15, ...]$$

**Numerical Conjecture A.**

$$A_{0,3} = [40, 1, 1, 1, 2, 3, 2, 1, 1, 6, 3] = \frac{127413}{3136} = \frac{3^4 \times 11^2 \times 13}{2^6 \times 7^2}$$

**Conjecture A.** *For every $n$, $k$ there is a rational number $A_{n,k}$ such that for $m \to \infty$*

$$|\mu_{n,m,k}| = \left| (A_{n,k} + o(1)) \, \frac{\mathrm{d}}{\mathrm{d}w} \tilde{\zeta}(w) \Big|_{w=\mathrm{w}_{n+k}} \mathrm{w}_{n+k}^{-m} \right|$$

**Numerical data.** For $n = 1,\ m = 511,\ k = 1$

$$\left| \frac{\mu_{n,m,k}}{\frac{\mathrm{d}}{\mathrm{d}w} \tilde{\zeta}(w) \Big|_{w=\mathrm{w}_{n+k}} \mathrm{w}_{n+k}^{-m}} \right| = 2.73913043478261054126841661225 3\ldots$$

$$= [2, 1, 2, 1, 5, 1024242810887, 2, 1, 1, \ldots]$$

**Conjecture A.** *For every $n$, $k$ there is a rational number $A_{n,k}$ such that for $m \to \infty$*

$$|\mu_{n,m,k}| = \left| (A_{n,k} + o(1)) \, \frac{\mathrm{d}}{\mathrm{d}w}\tilde{\zeta}(w)\Big|_{w=\mathrm{w}_{n+k}} \mathrm{w}_{n+k}^{-m} \right|$$

**Numerical data.** For $n = 1, \; m = 511, \; k = 1$

$$\left| \frac{\mu_{n,m,k}}{\frac{\mathrm{d}}{\mathrm{d}w}\tilde{\zeta}(w)\Big|_{w=\mathrm{w}_{n+k}} \mathrm{w}_{n+k}^{-m}} \right| = 2.7391304347826105412684166122 53\ldots$$

$$= [2, 1, 2, 1, 5, 1024242810887, 2, 1, 1, \ldots]$$

**Numerical Conjecture A.**

$$A_{1,1} = [2, 1, 2, 1, 5] = \frac{63}{23} = \frac{3^2 \times 7}{23}$$

**Conjecture A.** *For every $n$, $k$ there is a rational number $A_{n,k}$ such that for $m \to \infty$*

$$|\mu_{n,m,k}| = \left| (A_{n,k} + o(1)) \left. \frac{\mathrm{d}}{\mathrm{d}w} \tilde{\zeta}(w) \right|_{w=\mathrm{w}_{n+k}} \mathrm{w}_{n+k}^{-m} \right|$$

**Numerical data.** For $n = 1, \ m = 1023, \ k = 2$

$$\left| \frac{\mu_{n,m,k}}{\left. \frac{\mathrm{d}}{\mathrm{d}w} \tilde{\zeta}(w) \right|_{w=\mathrm{w}_{n+k}} \mathrm{w}_{n+k}^{-m}} \right| = 31.5026122291021708876609173 74\ldots$$

$$= [31, 1, 1, 95, 4, 1, 10, 2526428, 4, 1, 29, \ldots]$$

**Conjecture A.** *For every n, k there is a rational number $A_{n,k}$ such that for $m \to \infty$*

$$|\mu_{n,m,k}| = \left| (A_{n,k} + o(1)) \left. \frac{\mathrm{d}}{\mathrm{d}w} \tilde{\zeta}(w) \right|_{w=\mathrm{w}_{n+k}} \mathrm{w}_{n+k}^{-m} \right|$$

**Numerical data.** For $n = 1,\ m = 1023,\ k = 2$

$$\left| \frac{\mu_{n,m,k}}{\left. \frac{\mathrm{d}}{\mathrm{d}w} \tilde{\zeta}(w) \right|_{w=\mathrm{w}_{n+k}} \mathrm{w}_{n+k}^{-m}} \right| = 31.50261222910217088766091737 4...$$

$$= [31, 1, 1, 95, 4, 1, 10, 2526428, 4, 1, 29, ...]$$

**Numerical Conjecture A.**

$$A_{1,2} = [31, 1, 1, 95, 4, 1, 10] = \frac{325611}{10336} = \frac{3^2 \times 11^2 \times 13 \times 23}{2^5 \times 17 \times 19}$$

**Conjecture A.** *For every n, k there is a rational number $A_{n,k}$ such that for $m \to \infty$*

$$|\mu_{n,m,k}| = \left| (A_{n,k} + o(1)) \left. \frac{\mathrm{d}}{\mathrm{d}w} \tilde{\zeta}(w) \right|_{w=\mathrm{w}_{n+k}} \mathrm{w}_{n+k}^{-m} \right|$$

**Numerical data.** For $n = 2,\ m = 998,\ k = 1$

$$\left| \frac{\mu_{n,m,k}}{\left. \frac{\mathrm{d}}{\mathrm{d}w} \tilde{\zeta}(w) \right|_{w=\mathrm{w}_{n+k}} \mathrm{w}_{n+k}^{-m}} \right| = 13.6151598602901699517111494216649126\ldots$$

$$= [13, 1, 1, 1, 1, 2, 25, 1, 16, 2, 2, 334229, 19, 5, 1, \ldots$$

**Conjecture A.** *For every $n$, $k$ there is a rational number $A_{n,k}$ such that for $m \to \infty$*

$$|\mu_{n,m,k}| = \left| (A_{n,k} + o(1)) \left. \frac{\mathrm{d}}{\mathrm{d}w} \tilde{\zeta}(w) \right|_{w=\mathrm{w}_{n+k}} \mathrm{w}_{n+k}^{-m} \right|$$

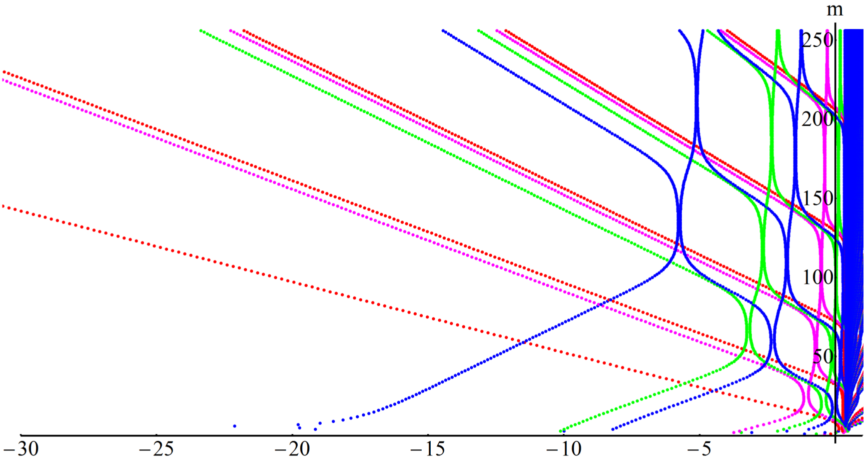
**Numerical data.** For $n = 2,\ m = 998,\ k = 1$

$$\left| \frac{\mu_{n,m,k}}{\left. \frac{\mathrm{d}}{\mathrm{d}w} \tilde{\zeta}(w) \right|_{w=\mathrm{w}_{n+k}} \mathrm{w}_{n+k}^{-m}} \right| = 13.6151598602901699517111494216649126 1\ldots$$

$$= [13, 1, 1, 1, 1, 2, 25, 1, 16, 2, 2, 334229, 19, 5, 1, \ldots$$

**Numerical Conjecture A.**

$$A_{2,1} = [13, 1, 1, 1, 1, 2, 25, 1, 16, 2, 2] = \frac{405405}{29776} = \frac{3^4 \times 5 \times 7 \times 11 \times 13}{2^4 \times 1861}$$




Values of $\log|\mu_{n,m,k}|$, for $n = 1, 2, 3, 4$

$$|\mu_{n,m,k}| = \left| (A_{n,k} + o(1)) \left. \frac{\mathrm{d}}{\mathrm{d}w} \tilde{\zeta}(w) \right|_{w=\mathrm{w}_{n+k}} \mathrm{w}_{n+k}^{-m} \right|$$

# RH: *For all n for m* $\longrightarrow \infty$

**Vers. 7:** $|\det(M_{n,m})|^{\frac{1}{m}} \to \prod_{j=1}^{n} \frac{2j+1}{2j}$

**Vers. 8:** $\left(\prod_{k=1}^{m} |\mu_{n,m,k}|\right)^{\frac{1}{m}} \to \prod_{j=1}^{n} \frac{2j+1}{2j}$

## RH: *For all n for m* $\longrightarrow \infty$

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**Vers. 9:** $\frac{1}{m} \sum_{k=1}^{m} \ln |\mu_{n,m,k}| \to \sum_{j=1}^{n} \ln \left(\frac{2j+1}{2j}\right)$

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**Vers. 10:** $\frac{1}{m} \sum_{k=1}^{m} \ln |\mu_{n+1,m,k}| - \frac{1}{m} \sum_{k=1}^{m} \ln |\mu_{n,m,k}| \to \ln \left(\frac{2n+3}{2n+2}\right)$

## RH: *For all n for m* $\longrightarrow \infty$

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**Vers. 11:** $\frac{1}{m} \sum_{k=1}^{m} \ln |\mu_{n+1,m,k}| - \frac{1}{m+1} \sum_{k=1}^{m+1} \ln |\mu_{n,m+1,k}| \to \ln\left(\frac{2n+3}{2n+2}\right)$

## RH: *For all n for m* $\longrightarrow \infty$

**Vers. 7:** $|\det(M_{n,m})|^{\frac{1}{m}} \to \prod_{j=1}^{n} \frac{2j+1}{2j}$

**Vers. 8:** $\left(\prod_{k=1}^{m} |\mu_{n,m,k}|\right)^{\frac{1}{m}} \to \prod_{j=1}^{n} \frac{2j+1}{2j}$

**Vers. 9:** $\frac{1}{m} \sum_{k=1}^{m} \ln |\mu_{n,m,k}| \to \sum_{j=1}^{n} \ln \left(\frac{2j+1}{2j}\right)$

**Vers. 10:** $\frac{1}{m} \sum_{k=1}^{m} \ln |\mu_{n+1,m,k}| - \frac{1}{m} \sum_{k=1}^{m} \ln |\mu_{n,m,k}| \to \ln \left(\frac{2n+3}{2n+2}\right)$

**Vers. 11:** $\frac{1}{m} \sum_{k=1}^{m} \ln |\mu_{n+1,m,k}| - \frac{1}{m+1} \sum_{k=1}^{m+1} \ln |\mu_{n,m+1,k}| \to \ln \left(\frac{2n+3}{2n+2}\right)$

**Vers. 12:** $\frac{1}{m} \sum_{k=1}^{m} \ln |\mu_{n+1,m,k}| - \frac{1}{m} \sum_{k=1}^{m+1} \ln |\mu_{n,m+1,k}| \to \ln \left(\frac{2n+3}{2n+2}\right)$

## RH: *For all n for m* $\longrightarrow \infty$

**Vers. 7:** $|\det(M_{n,m})|^{\frac{1}{m}} \to \prod_{j=1}^{n} \frac{2j+1}{2j}$

**Vers. 8:** $\left(\prod_{k=1}^{m} |\mu_{n,m,k}|\right)^{\frac{1}{m}} \to \prod_{j=1}^{n} \frac{2j+1}{2j}$

**Vers. 9:** $\frac{1}{m}\sum_{k=1}^{m} \ln|\mu_{n,m,k}| \to \sum_{j=1}^{n} \ln\left(\frac{2j+1}{2j}\right)$

**Vers. 10:** $\frac{1}{m}\sum_{k=1}^{m} \ln|\mu_{n+1,m,k}| - \frac{1}{m}\sum_{k=1}^{m} \ln|\mu_{n,m,k}| \to \ln\left(\frac{2n+3}{2n+2}\right)$

**Vers. 11:** $\frac{1}{m}\sum_{k=1}^{m} \ln|\mu_{n+1,m,k}| - \frac{1}{m+1}\sum_{k=1}^{m+1} \ln|\mu_{n,m+1,k}| \to \ln\left(\frac{2n+3}{2n+2}\right)$

**Vers. 12:** $\frac{1}{m}\sum_{k=1}^{m} \ln|\mu_{n+1,m,k}| - \frac{1}{m}\sum_{k=1}^{m+1} \ln|\mu_{n,m+1,k}| \to \ln\left(\frac{2n+3}{2n+2}\right)$

**Vers. 13:**

$$\frac{1}{m}\sum_{k=1}^{m}(\ln|\mu_{n+1,m,k}| - \ln|\mu_{n,m+1,k+1}|) - \frac{1}{m}\ln|\mu_{n,m+1,1}| \to \ln\left(\frac{2n+3}{2n+2}\right)$$

# RH (version 13)

$$\frac{1}{m}\sum_{k=1}^{m}(\ln|\mu_{n+1,m,k}| - \ln|\mu_{n,m+1,k+1}|) - \frac{1}{m}\ln|\mu_{n,m+1,1}| \to \ln\left(\frac{2n+3}{2n+2}\right)$$

## RH (version 13)

$$\frac{1}{m}\sum_{k=1}^{m}(\ln|\mu_{n+1,m,k}| - \ln|\mu_{n,m+1,k+1}|) - \frac{1}{m}\ln|\mu_{n,m+1,1}| \to \ln\left(\frac{2n+3}{2n+2}\right)$$

**Conjecture A.** $|\mu_{n,m,1}| = \left|(A_{n,1}+o(1))\ \frac{\mathrm{d}}{\mathrm{d}w}\tilde{\zeta}(w)\Big|_{w=\mathrm{w}_{n+1}}\mathrm{w}_{n+1}^{-m}\right|$

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**Conjecture B.** *For all n*

$$B_{n,m} = \sum_{k=1}^{m}(\ln|\mu_{n+1,m,k}| - \ln|\mu_{n,m+1,k+1}|) = o(m)$$

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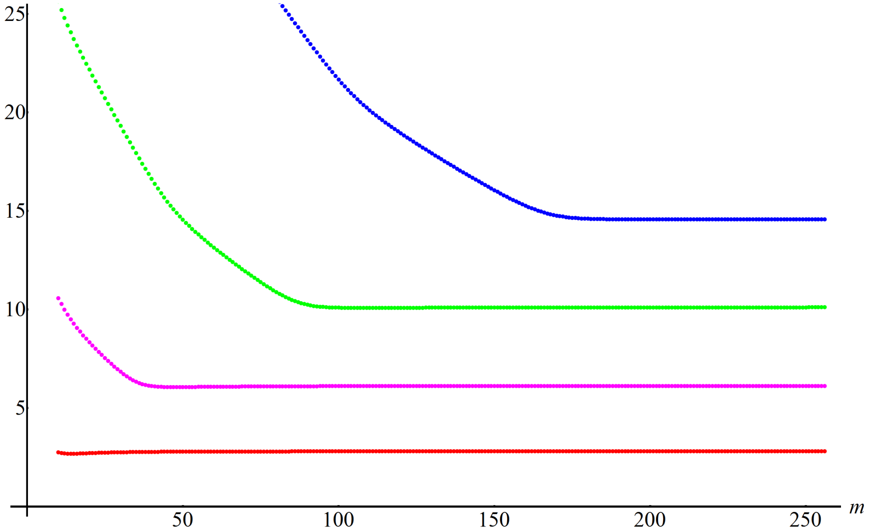
$$\textbf{A} \implies (\textbf{B} \iff \textbf{RH})$$




Values of $-B_{n,m}$ for $n = 1, 2, 3, 4$

**Conjecture B (stronger form).** *For every $n$ there exists a number $B_n$ such that for $m \to \infty$*

$$B_{n,m} = \sum_{k=1}^{m} (\ln |\mu_{n+1,m,k}| - \ln |\mu_{n,m+1,k+1}|) \to B_n.$$

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**Numerical data.**

$$\begin{aligned} e^{B_{0,127}} &= 0.5555555555695406424306420... \\ e^{B_{0,511}} &= 0.5555555555555555555555555555555555555555610... \end{aligned}$$

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**Numerical data.**

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**Numerical Conjecture B.** $B_0 = \ln\left(\frac{5}{9}\right)$

**Conjecture B (stronger form).** *For every $n$ there exists a number $B_n$ such that for $m \to \infty$*

$$B_{n,m} = \sum_{k=1}^{m} (\ln |\mu_{n+1,m,k}| - \ln |\mu_{n,m+1,k+1}|) \to B_n.$$

**Numerical data.**

$$\begin{aligned} e^{B_{1,498}} &= 0.060869565217391334297063067299596\ldots \\ &= [0, 16, 2, 2, 1, 2524751023850, 3, 20, 1, 1, 3, 11, 3, \ldots] \\ e^{B_{1,998}} &= [0, 16, 2, 2, 1, 2420082889472848925180620903, 3, 2, 4, 3, 5, 1, \ldots] \end{aligned}$$

**Conjecture B (stronger form).** *For every $n$ there exists a number $B_n$ such that for $m \to \infty$*

$$B_{n,m} = \sum_{k=1}^{m} (\ln|\mu_{n+1,m,k}| - \ln|\mu_{n,m+1,k+1}|) \to B_n.$$

**Numerical data.**

$$\begin{aligned} e^{B_{1,498}} &= 0.0608695652173913342970630672995 96\ldots \\ &= [0, 16, 2, 2, 1, 2524751023850, 3, 20, 1, 1, 3, 11, 3, \ldots] \\ e^{B_{1,998}} &= [0, 16, 2, 2, 1, 242008288947284892518062 0903, 3, 2, 4, 3, 5, 1, \ldots] \end{aligned}$$

$$[0, 16, 2, 2, 1] = \frac{7}{115} = \frac{7}{5 \times 23}$$

**Conjecture B (stronger form).** *For every n there exists a number $B_n$ such that for $m \to \infty$*

$$B_{n,m} = \sum_{k=1}^{m} (\ln |\mu_{n+1,m,k}| - \ln |\mu_{n,m+1,k+1}|) \to B_n.$$

**Numerical data.**

$$\begin{aligned} e^{B_{1,498}} &= 0.060869565217391334297063067299596... \\ &= [0, 16, 2, 2, 1, 2524751023850, 3, 20, 1, 1, 3, 11, 3, ...] \\ e^{B_{1,998}} &= [0, 16, 2, 2, 1, 242008288947284892518062090 3, 3, 2, 4, 3, 5, 1, ...] \end{aligned}$$

$$[0, 16, 2, 2, 1] = \frac{7}{115} = \frac{7}{5 \times 23}$$

**Numerical Conjecture B.** $B_1 = \ln\left(\frac{7}{115}\right)$

**Conjecture B (stronger form).** *For every $n$ there exists a number $B_n$ such that for $m \to \infty$*

$$B_{n,m} = \sum_{k=1}^{m} (\ln|\mu_{n+1,m,k}| - \ln|\mu_{n,m+1,k+1}|) \to B_n.$$

**Numerical data.**

$$\begin{aligned} e^{B_{2,897}} &= 0.002195440239502577525797480459737793 65... \\ &= [0, 455, 2, 23, 3, 39656499, 38, 14, 2, 1, 1, 3, 1, 7, , ...] \\ e^{B_{2,997}} &= 0.00219544023950257173864042772451897584... \\ &= [0, 455, 2, 23, 3, 1505709159, 2, 1, 1, 1, 3, 8, ...] \end{aligned}$$

**Conjecture B (stronger form).** *For every n there exists a number $B_n$ such that for $m \to \infty$*

$$B_{n,m} = \sum_{k=1}^{m} (\ln|\mu_{n+1,m,k}| - \ln|\mu_{n,m+1,k+1}|) \to B_n.$$

**Numerical data.**

$$\begin{aligned} e^{B_{2,897}} &= 0.002195440239502577525797480459737793 65... \\ &= [0, 455, 2, 23, 3, 39656499, 38, 14, 2, 1, 1, 3, 1, 7, , ...] \\ e^{B_{2,997}} &= 0.002195440239502571738640427724518975 84... \\ &= [0, 455, 2, 23, 3, 1505709159, 2, 1, 1, 1, 3, 8, ...] \end{aligned}$$

$$[0, 455, 2, 23, 3] = \frac{143}{65135} = \frac{11 \times 13}{5 \times 7 \times 1861}$$

**Conjecture B (stronger form).** *For every n there exists a number $B_n$ such that for $m \to \infty$*

$$B_{n,m} = \sum_{k=1}^{m} (\ln |\mu_{n+1,m,k}| - \ln |\mu_{n,m+1,k+1}|) \to B_n.$$

**Numerical data.**

$$\begin{aligned} e^{B_{2,897}} &= 0.00219544023950257752579748045973779365... \\ &= [0, 455, 2, 23, 3, 39656499, 38, 14, 2, 1, 1, 3, 1, 7, , ...] \\ e^{B_{2,997}} &= 0.00219544023950257173864042772451897584... \\ &= [0, 455, 2, 23, 3, 1505709159, 2, 1, 1, 1, 3, 8, ...] \end{aligned}$$

$$[0, 455, 2, 23, 3] = \frac{143}{65135} = \frac{11 \times 13}{5 \times 7 \times 1861}$$

**Numerical Conjecture B.** $B_2 = \ln\left(\frac{143}{65135}\right)$

**Conjecture B (stronger form).** *For every $n$ there exists a number $B_n$ such that for $m \to \infty$*

$$B_{n,m} = \sum_{k=1}^{m} (\ln|\mu_{n+1,m,k}| - \ln|\mu_{n,m+1,k+1}|) \to B_n.$$

**Numerical data.**

$$\begin{aligned} e^{B_{3,1796}} &= 0.000040591507258173924884118015229633 7\ldots \\ &= [0, 24635, 1, 2, 3, 1, 1, 3, 2, 1082513488, 1, 2, 4, 3, 2, 3, , , \ldots] \\ e^{B_{3,1996}} &= 0.0000405915072581739248410776752564941286\ldots \\ &= [0, 24635, 1, 2, 3, 1, 1, 3, 2, 96917181255, 273, 1, 1, 3, 2, , \ldots] \end{aligned}$$

**Conjecture B (stronger form).** *For every $n$ there exists a number $B_n$ such that for $m \to \infty$*

$$B_{n,m} = \sum_{k=1}^{m} (\ln|\mu_{n+1,m,k}| - \ln|\mu_{n,m+1,k+1}|) \to B_n.$$

**Numerical data.**

$$\begin{aligned} e^{B_{3,1796}} &= 0.0000405915072581739248841180152296337\ldots \\ &= [0, 24635, 1, 2, 3, 1, 1, 3, 2, 1082513488, 1, 2, 4, 3, 2, 3, , , \ldots] \\ e^{B_{3,1996}} &= 0.0000405915072581739248410776752564941286\ldots \\ &= [0, 24635, 1, 2, 3, 1, 1, 3, 2, 96917181255, 273, 1, 1, 3, 2, , \ldots] \end{aligned}$$

$$[0, 24635, 1, 2, 3, 1, 1, 3, 2] = \frac{187}{4606875} = \frac{11 \times 17}{3^4 \times 5^4 \times 7 \times 13}$$

**Conjecture B (stronger form).** *For every n there exists a number $B_n$ such that for $m \to \infty$*

$$B_{n,m} = \sum_{k=1}^{m} (\ln|\mu_{n+1,m,k}| - \ln|\mu_{n,m+1,k+1}|) \to B_n.$$

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**Numerical Conjecture B.** $B_3 = \ln\left(\frac{187}{4606875}\right)$

**Conjecture B (stronger form).** *For every $n$ there exists a number $B_n$ such that for $m \to \infty$*

$$B_{n,m} = \sum_{k=1}^{m} (\ln|\mu_{n+1,m,k}| - \ln|\mu_{n,m+1,k+1}|) \to B_n.$$

**Numerical data.**

$$\begin{aligned} e^{B_{4,3996}} &= .000000460297681987078674528702352178315230\ldots \\ &= [0, 2172507, 7, 4, 1, 1, 1, 9, 1, 2, 1, 24667496266172, 2, 4, 4, \ldots] \end{aligned}$$

**Conjecture B (stronger form).** *For every $n$ there exists a number $B_n$ such that for $m \to \infty$*

$$B_{n,m} = \sum_{k=1}^{m} (\ln|\mu_{n+1,m,k}| - \ln|\mu_{n,m+1,k+1}|) \to B_n.$$

**Numerical data.**

$$\begin{aligned} e^{B_{4,3996}} &= .0000004602976819870786745287023521783152 30\ldots \\ &= [0, 2172507, 7, 4, 1, 1, 1, 9, 1, 2, 1, 24667496266172, 2, 4, 4, \ldots] \end{aligned}$$

$$\begin{aligned} [0, 2172507, 7, 4, 1, 1, 1, 9, 1, 2, 1] &= \frac{4199}{9122357475} \\ &= \frac{17 \times 19}{3^2 \times 5^2 \times 7 \times 11 \times 526543} \end{aligned}$$

**Conjecture B (stronger form).** *For every $n$ there exists a number $B_n$ such that for $m \to \infty$*

$$B_{n,m} = \sum_{k=1}^{m} (\ln|\mu_{n+1,m,k}| - \ln|\mu_{n,m+1,k+1}|) \to B_n.$$

**Numerical data.**

$$\begin{aligned} e^{B_{4,3996}} &= .00000046029768198707867452870235217831523 0\ldots \\ &= [0, 2172507, 7, 4, 1, 1, 1, 9, 1, 2, 1, 24667496266172, 2, 4, 4, \ldots] \end{aligned}$$

$$\begin{aligned} [0, 2172507, 7, 4, 1, 1, 1, 9, 1, 2, 1] &= \frac{4199}{9122357475} \\ &= \frac{17 \times 19}{3^2 \times 5^2 \times 7 \times 11 \times 526543} \end{aligned}$$

**Numerical Conjecture B.** $B_4 = \ln\left(\frac{4199}{9122357475}\right)$

**Conjecture B (stronger form).** *For every n there exists a number $B_n$ such that for $m \to \infty$*

$$B_{n,m} = \sum_{k=1}^{m} (\ln |\mu_{n+1,m,k}| - \ln |\mu_{n,m+1,k+1}|) \to B_n.$$

**Numerical data.**

$$e^{B_{5,3996}} = .00000000353461251528901387816997037638317186 20\ldots$$

**Conjecture B (stronger form).** *For every $n$ there exists a number $B_n$ such that for $m \to \infty$*

$$B_{n,m} = \sum_{k=1}^{m} (\ln|\mu_{n+1,m,k}| - \ln|\mu_{n,m+1,k+1}|) \to B_n.$$

**Numerical data.**

$$e^{B_{5,3996}} = .00000000353461251528901387816997037638317118620\ldots$$

$$\frac{185725}{52544656365201} = \frac{5^2 \times 17 \times 19 \times 23}{3^2 \times 7 \times 11 \times 13 \times 19793 \times 294673}$$

**Conjecture B (stronger form).** *For every n there exists a number $B_n$ such that for $m \to \infty$*

$$B_{n,m} = \sum_{k=1}^{m} (\ln|\mu_{n+1,m,k}| - \ln|\mu_{n,m+1,k+1}|) \to B_n.$$

**Numerical data.**

$$e^{B_{5,3996}} = .0000000035346125152890138781699703763831718620\ldots$$

$$\frac{185725}{52544656365201} = \frac{5^2 \times 17 \times 19 \times 23}{3^2 \times 7 \times 11 \times 13 \times 19793 \times 294673}$$

**Numerical Conjecture.** $B_5 = \ln\left(\frac{185725}{52544656365201}\right)$

**Numerical Conjectures B.**

$$
\begin{aligned}
B_0 = \ln\left(\frac{5}{9}\right) &= \ln\left(\frac{5}{3^2}\right) \\
B_1 = \ln\left(\frac{7}{115}\right) &= \ln\left(\frac{7}{5\times 23}\right) \\
B_2 = \ln\left(\frac{143}{65135}\right) &= \ln\left(\frac{11\times 13}{5\times 7\times 1861}\right) \\
B_3 = \ln\left(\frac{187}{4606875}\right) &= \ln\left(\frac{11\times 17}{3^4\times 5^4\times 7\times 13}\right) \\
B_4 = \ln\left(\frac{4199}{9122357475}\right) &= \ln\left(\frac{17\times 19}{3^2\times 5^2\times 7\times 11\times 526543}\right) \\
B_5 = \ln\left(\frac{185725}{52544656365201}\right) &= \ln\left(\frac{5^2\times 17\times 19\times 23}{3^2\times 7\times 11\times 13\times 19793\times 294673}\right) \\
B_6 &= ?
\end{aligned}
$$

**Numerical Conjectures B for odd characters.**

$$\tilde{B}_0 = \ln\left(\frac{3}{4}\right) = \ln\left(\frac{3}{2^2}\right)$$

$$\tilde{B}_1 = \ln\left(\frac{105}{704}\right) = \ln\left(\frac{3 \times 5 \times 7}{2^6 \times 11}\right)$$

$$\tilde{B}_2 = \ln\left(\frac{385}{46848}\right) = \ln\left(\frac{5 \times 7 \times 11}{2^8 \times 3 \times 61}\right)$$

$$\tilde{B}_3 = \ln\left(\frac{45045}{220512256}\right) = \ln\left(\frac{3^2 \times 5 \times 7 \times 11 \times 13}{2^{14} \times 43 \times 313}\right)$$

$$\tilde{B}_4 = ?$$