

Eigenvalues and eigenvectors
of some Hankel matrices
related to the zeta and L -functions

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[http://logic.pdmi.ras.ru/~yumat/personaljournal/
zetahiddenlife/zetahiddenlife.html](http://logic.pdmi.ras.ru/~yumat/personaljournal/zetahiddenlife/zetahiddenlife.html)

Classical Statement of Riemann's Hypothesis

The Riemann's zeta function:

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

Riemann's Hypothesis (version 1). *All non-trivial zeros of function $\zeta(z)$ lie on the critical line $\operatorname{Re}(z) = \frac{1}{2}$.*

First Reformulation

Tradition:

$$\xi(z) = \pi^{-\frac{z}{2}}(z-1)\zeta(z)\Gamma(1+\frac{z}{2})$$

Functional equation:

$$\xi(1-z) = \xi(z)$$

Our choice:

$$\zeta^*(z) = 2(z-1)\zeta(z) \quad \zeta^*(0) = 1$$

RH (version 2) *The trivial zeros $z_1 = -2, z_2 = -4, \dots, z_n = -2n, \dots$ are the only zeros of function $\zeta^*(z)$ lying in the half-plane $\operatorname{Re}(z) < \frac{1}{2}$.*

.

Change of the variable

$$z = \frac{w}{w+1} \quad w = \frac{z}{1-z}$$

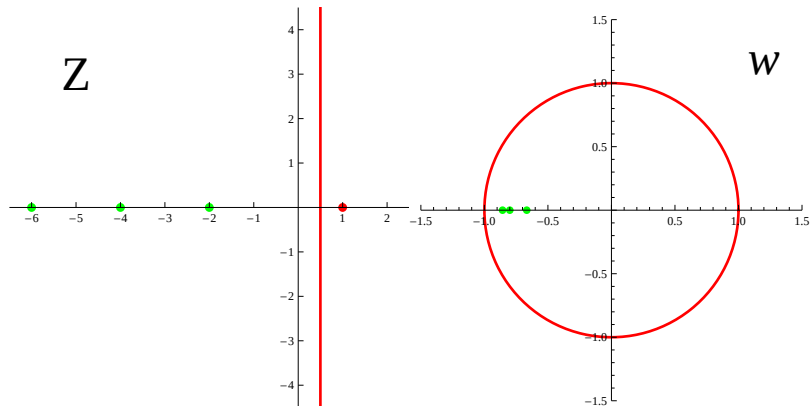
Under this transformation the critical line becomes the critical circle $|w| = 1$, the half-plane $\operatorname{Re}(z) < \frac{1}{2}$ becomes the interior of this circle, and points

$$w_1 = \frac{z_1}{1-z_1} = -\frac{2}{3}, w_2 = \frac{z_2}{1-z_2} = -\frac{4}{5}, \dots, w_n = \frac{z_n}{1-z_n} = -\frac{2n}{2n+1}, \dots$$

become the trivial zeros of the function

$$\tilde{\zeta}(w) = \zeta^*\left(\frac{w}{w+1}\right)$$

Change of the variable

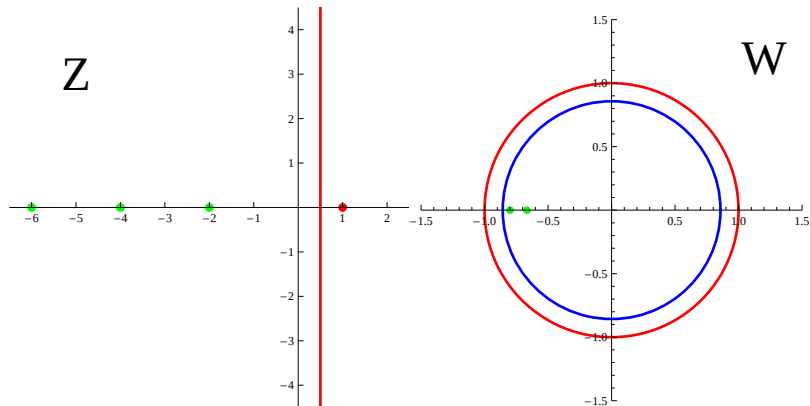


RH (version 3) *The trivial zeros*

$$w_1 = -\frac{2}{3}, w_2 = -\frac{4}{5}, \dots, w_n = -\frac{2n}{2n+1}, \dots$$

are the only zeros of function $\tilde{\zeta}(w) = \zeta^*\left(\frac{w}{w+1}\right)$ lying in the open disk $|w| < 1$.

First parameter



RH (version 4). For every n the trivial zeros

$$w_1 = -\frac{2}{3}, w_2 = -\frac{4}{5}, \dots, w_n = -\frac{2n}{2n+1}$$

are the only zeros of function $\zeta(w)$ lying in the closed disk $|w| \leq \frac{2n+1}{2n+2}$.

Padé approximations

$$\begin{aligned}\tilde{\zeta}(w) &\approx \frac{P_{n,m}(w)}{Q_{n,m}(w)} = \frac{1 + p_{n,m,1}w + \cdots + p_{n,m,n}w^n}{1 + q_{n,m,1}w + \cdots + q_{n,m,m}w^m} \\ &= \tilde{\zeta}(w) + O(w^{n+m+1})\end{aligned}$$

Theorem (de Montessus de Ballore, [1905]). *For $m \rightarrow \infty$*

$$P_{n,m}(w) \rightarrow \prod_{j=1}^n \left(1 - \frac{w}{v_j}\right)$$

where $v_1, \dots, v_n$ are zeroes of $\tilde{\zeta}$ with least absolute values (provided that the choice of such zeroes is unique).

RH (version 5). *For every n for $m \rightarrow \infty$*

$$P_{n,m}(w) \rightarrow \prod_{j=1}^n \left(1 - \frac{w}{w_j}\right)$$

Another reformulations

$$\tilde{\zeta}(w) \approx \frac{P_{n,m}(w)}{Q_{n,m}(w)} = \frac{1 + p_{n,m,1}w + \cdots + p_{n,m,n}w^n}{1 + q_{n,m,1}w + \cdots + q_{n,m,m}w^m}$$

RH (version 5). For every n for $m \rightarrow \infty$

$$P_{n,m}(w) \rightarrow \prod_{j=1}^n \left(1 - \frac{w}{w_j}\right)$$

RH (version 6). For every n for $m \rightarrow \infty$

$$p_{n,m,n} \rightarrow \prod_{j=1}^n \left(-\frac{1}{w_j}\right) = \prod_{j=1}^n \frac{2j+1}{2j}$$

Determinants

$$\tilde{\zeta}(w) = \sum_{j=-\infty}^{\infty} \theta_j w^j \approx \frac{1 + p_{n,m,1}w + \cdots + p_{n,m,n}w^n}{1 + q_{n,m,1}w + \cdots + q_{n,m,m}w^m}$$

$$p_{n,m,n} = (-1)^{n+m+1} \frac{\det(M_{n,m+1})}{\det(M_{n,m})}$$

$$M_{n,m} = (-1)^{n+m} \begin{pmatrix} \theta_{n+m-1} & \theta_{n+m-2} & \cdots & \theta_n \\ \theta_{n+m-2} & \theta_{n+m-3} & \cdots & \theta_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_n & \theta_{n-1} & \cdots & \theta_{n-m+1} \end{pmatrix}$$

RH (version 7). For all n for $m \rightarrow \infty$

$$|\det(M_{n,m})|^{\frac{1}{m}} \rightarrow \prod_{j=1}^n \frac{2j+1}{2j}$$

Eigenvalues

$$M_{n,m} = (-1)^{n+m} \begin{pmatrix} \theta_{n+m-1} & \theta_{n+m-2} & \cdots & \theta_n \\ \theta_{n+m-2} & \theta_{n+m-3} & \cdots & \theta_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_n & \theta_{n-1} & \cdots & \theta_{n-m+1} \end{pmatrix} \quad \tilde{\zeta}(w) = \sum_{j=-\infty}^{\infty} \theta_j w^j$$

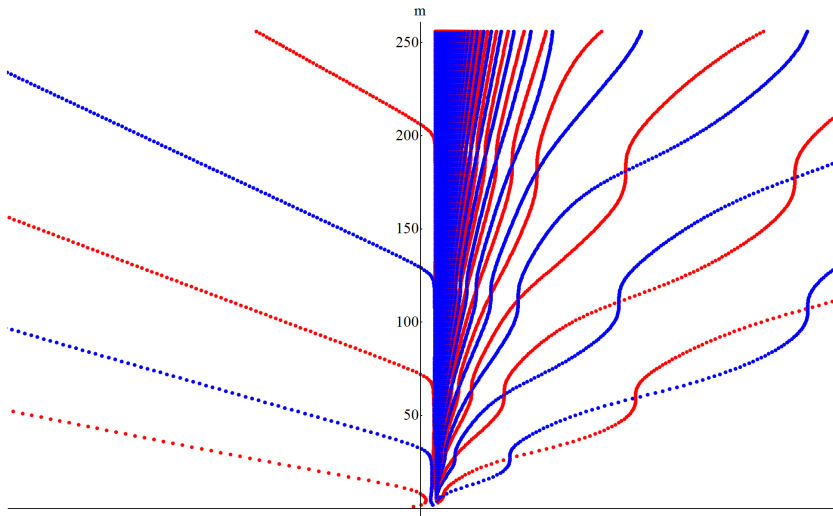
Let $\mu_{n,m,1}, \mu_{n,m,2}, \dots, \mu_{n,m,m}$ be the eigenvalues of the matrix $M_{n,m}$

$$|\mu_{n,m,1}| \leq |\mu_{n,m,2}| \leq \cdots \leq |\mu_{n,m,m}|$$

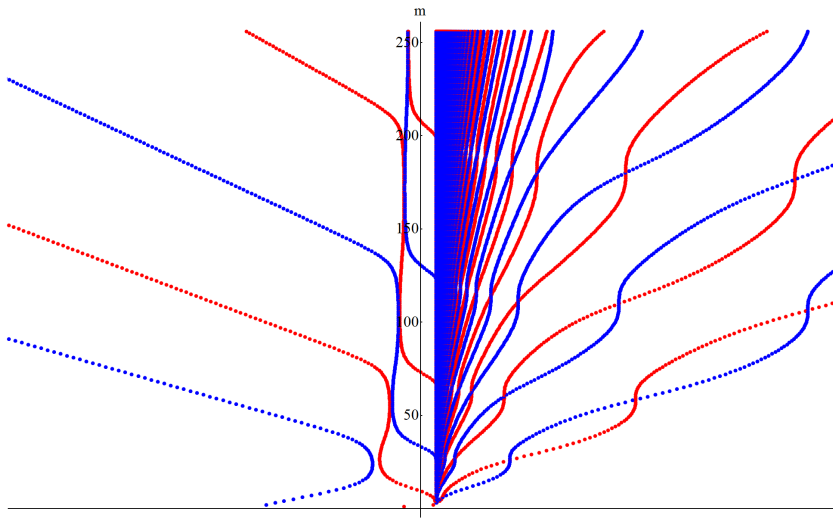
$$\det(M_{n,m}) = \mu_{n,m,1} \mu_{n,m,2} \cdots \mu_{n,m,m}$$

RH (version 8) For all n for $m \rightarrow \infty$

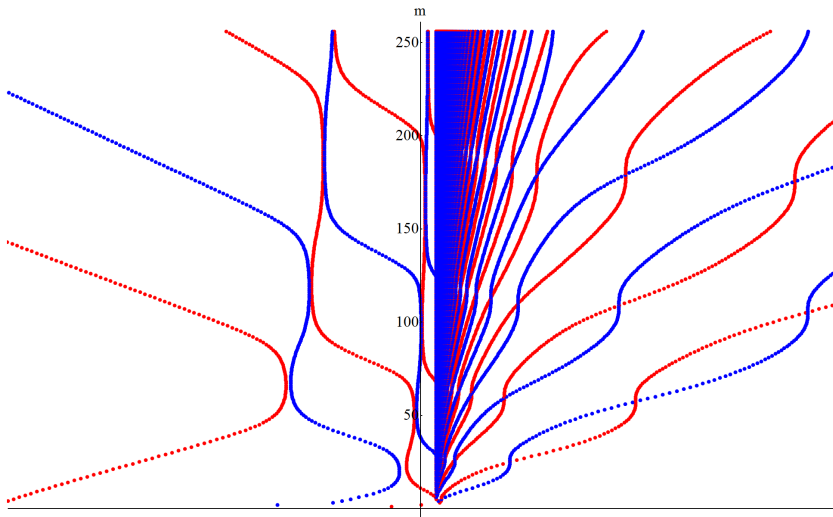
$$\left(\prod_{k=1}^m |\mu_{n,m,k}| \right)^{\frac{1}{m}} \rightarrow \prod_{j=1}^n \frac{2j+1}{2j}$$



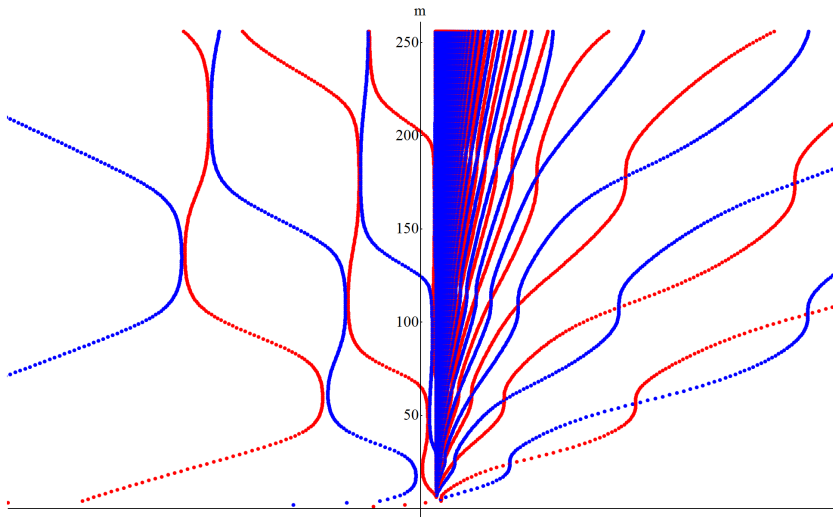
Values of $\log |\mu_{1,m,k}|$, red for $\mu_{1,m,k} > 0$ and blue for $\mu_{1,m,k} < 0$ for $m = 1 \dots 256$



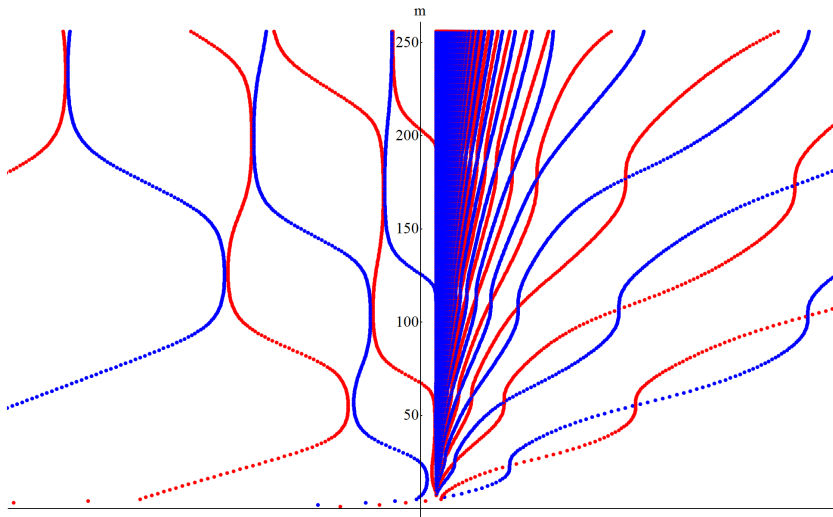
Values of $\log |\mu_{2,m,k}|$, red for $\mu_{2,m,k} > 0$ and blue for $\mu_{2,m,k} < 0$ for $m = 1 \dots 256$



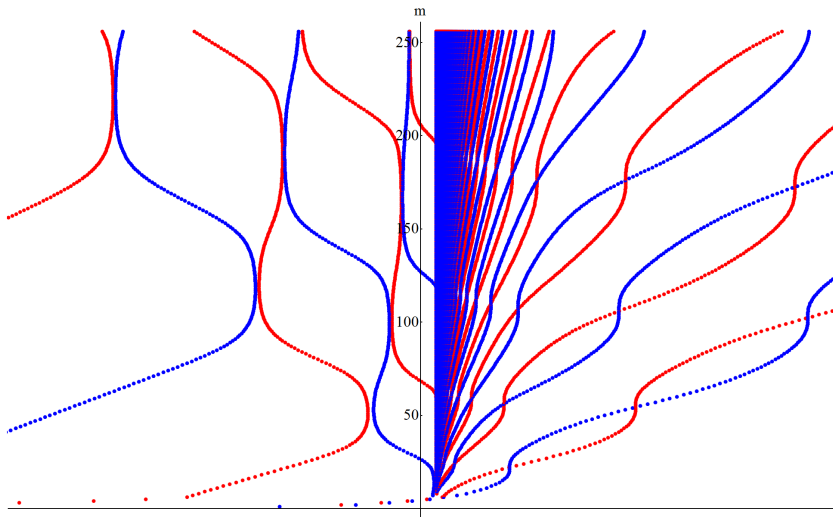
Values of $\log |\mu_{3,m,k}|$, red for $\mu_{3,m,k} > 0$ and blue for $\mu_{3,m,k} < 0$ for $m = 1 \dots 256$



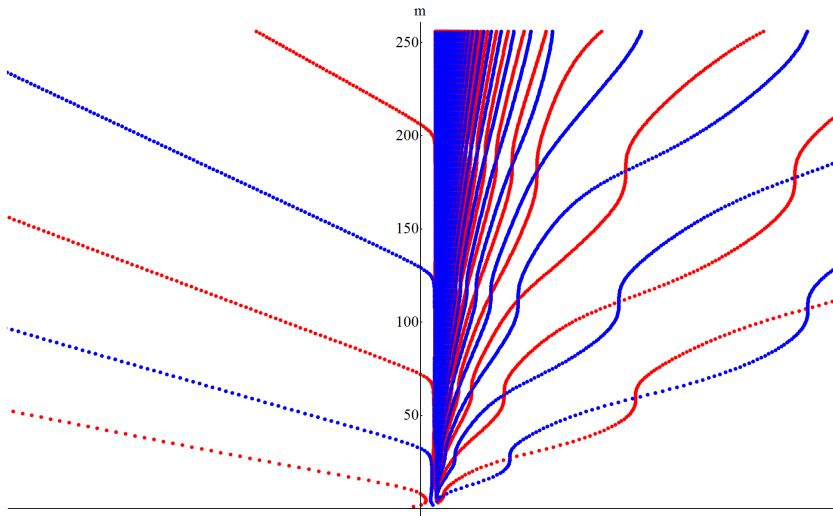
Values of $\log |\mu_{4,m,k}|$, red for $\mu_{4,m,k} > 0$ and blue for $\mu_{4,m,k} < 0$ for $m = 1 \dots 256$



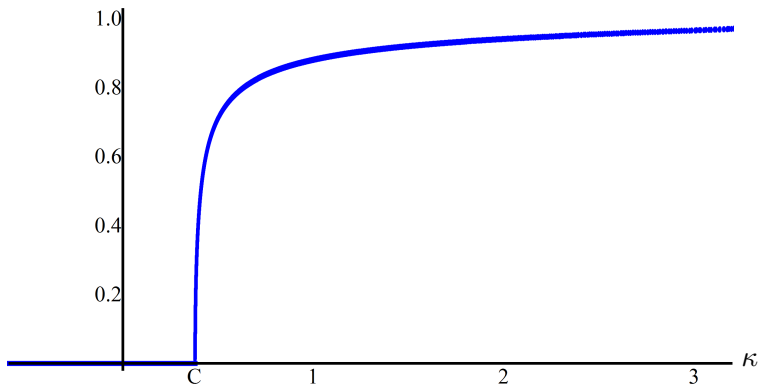
Values of $\log |\mu_{5,m,k}|$, red for $\mu_{5,m,k} > 0$ and blue for $\mu_{5,m,k} < 0$ for $m = 1 \dots 256$



Values of $\log |\mu_{6,m,k}|$, red for $\mu_{6,m,k} > 0$ and blue for $\mu_{6,m,k} < 0$ for $m = 1 \dots 256$

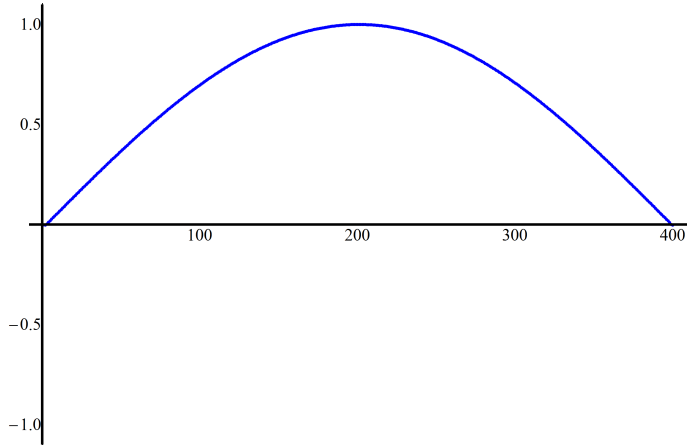


Values of $\log |\mu_{1,m,k}|$, red for $\mu_{1,m,k} > 0$ and blue for $\mu_{1,m,k} < 0$ for $m = 1 \dots 256$



Limiting percentage of eigenvalues $\mu_{n,m,k}$ such that $\ln |\mu_{n,m,k}| < \kappa$

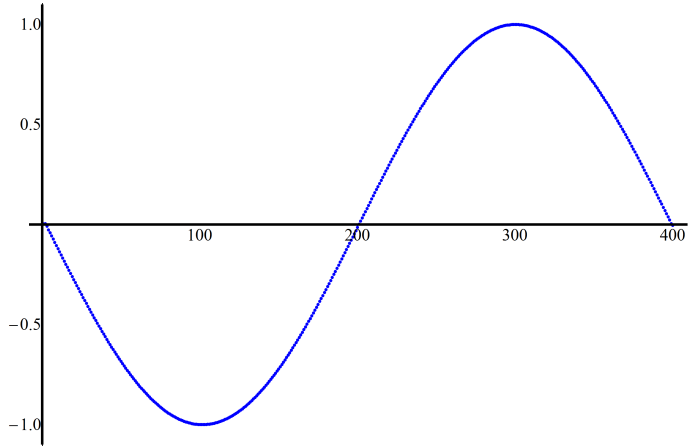
$$C = 0.378679220\dots = \ln \left(-\zeta \left(\frac{1}{2} \right) \right)$$



Entries of the normalized eigenvector for the eigenvalue

$$\mu_{1,400,9} = -1.4603552\dots$$

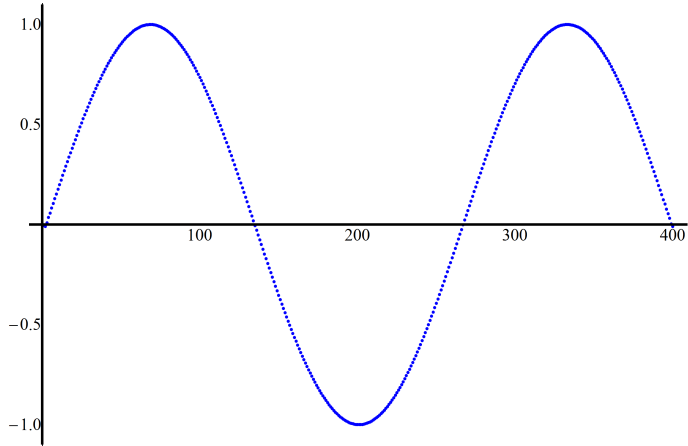
$$\zeta(.5) = -1.4603545\dots$$



Entries of the normalized eigenvector for the eigenvalue

$$\mu_{1,400,10} = 1.4603574\dots$$

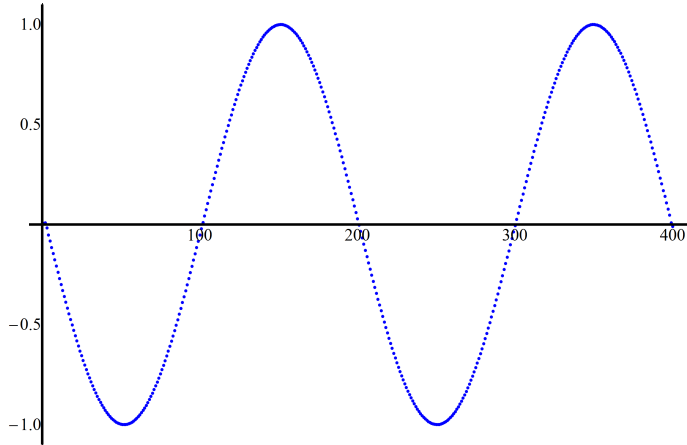
$$-\zeta(.5) = 1.4603545\dots$$



Entries of the normalized eigenvector for the eigenvalue

$$\mu_{1,400,11} = -1.4603610\dots$$

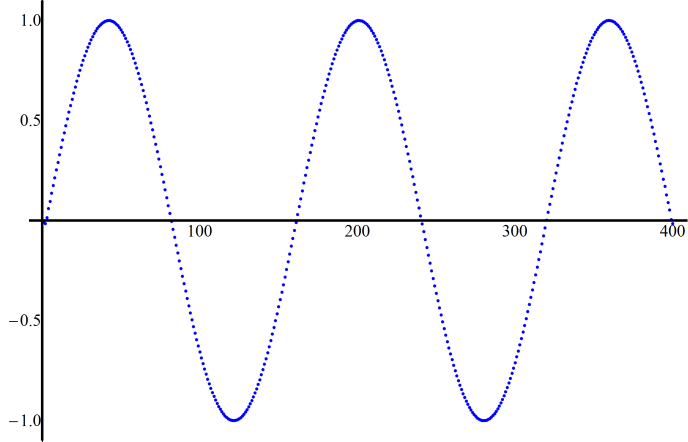
$$\zeta(.5) = -1.4603545\dots$$



Entries of the normalized eigenvector for the eigenvalue

$$\mu_{1,400,12} = 1.4603660\dots$$

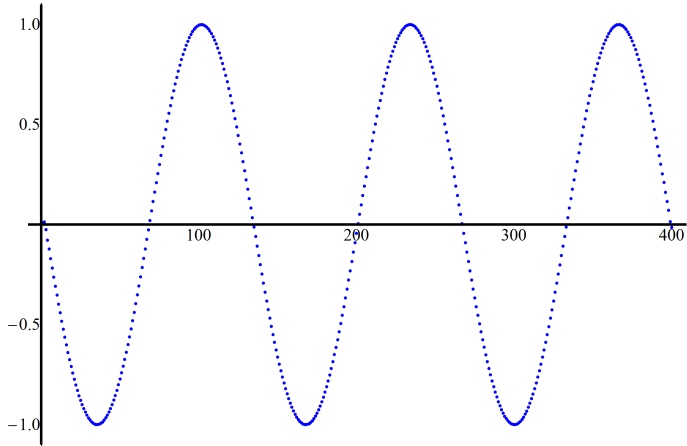
$$-\zeta(.5) = 1.4603545\dots$$



Entries of the normalized eigenvector for the eigenvalue

$$\mu_{1,400,13} = -1.4603726\dots$$

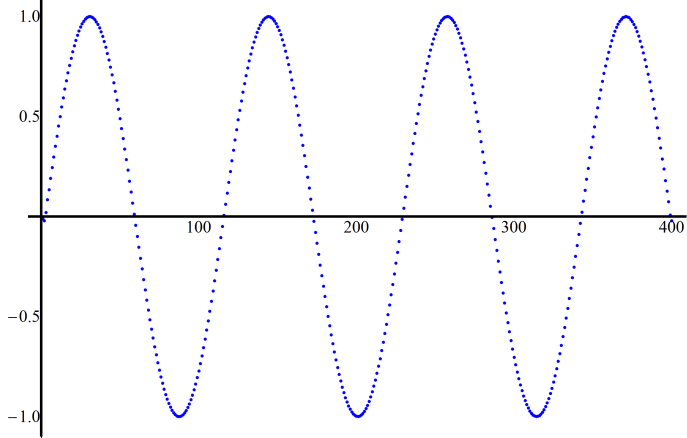
$$\zeta(.5) = -1.4603545\dots$$



Entries of the normalized eigenvector for the eigenvalue

$$\mu_{1,400,14} = 1.4603805\dots$$

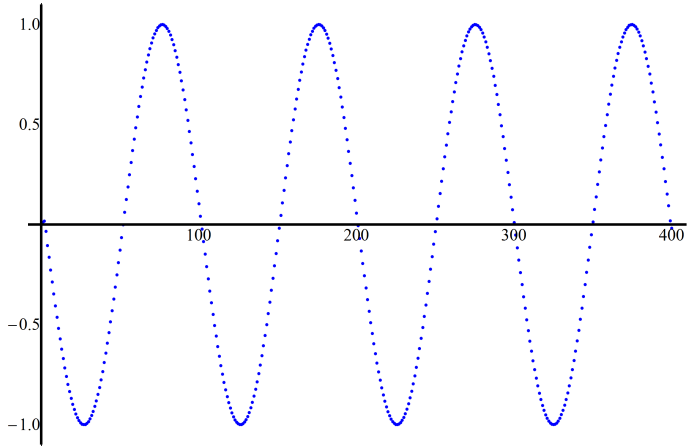
$$-\zeta(.5) = 1.4603545\dots$$



Entries of the normalized eigenvector for the eigenvalue

$$\mu_{1,400,15} = -1.4603900\dots$$

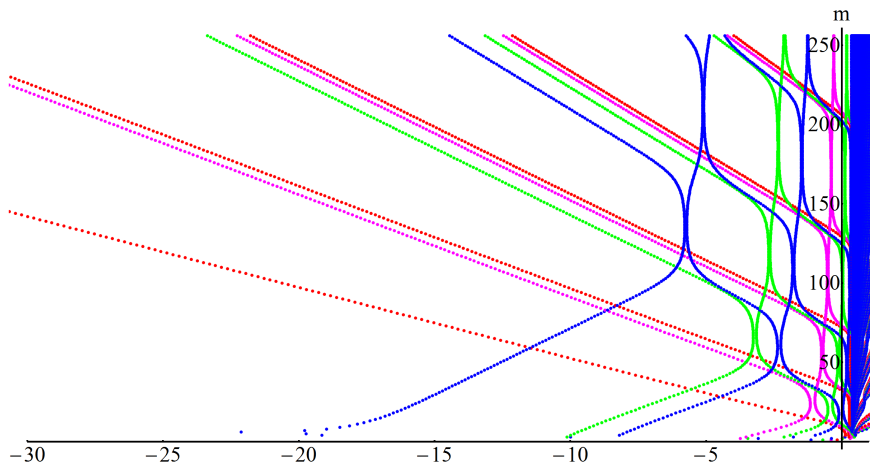
$$\zeta(.5) = -1.4603545\dots$$



Entries of the normalized eigenvector for the eigenvalue

$$\mu_{1,400,16} = 1.4604008\dots$$

$$-\zeta(.5) = 1.4603545\dots$$



Values of $\log |\mu_{n,m,k}|$ for $n = 1, 2, 3, 4$ for $m = 1, \dots, 256$

Conjecture A. *For every n, k there is a rational number $A_{n,k}$ such that for $m \rightarrow \infty$*

$$|\mu_{n,m,k}| = \left| (A_{n,k} + o(1)) \left. \frac{d}{dw} \tilde{\zeta}(w) \right|_{w=w_{n+k}} w_{n+k}^{-m} \right|$$

$w_n = -\frac{2n}{2n+1}$ is a trivial zero of $\zeta\left(\frac{w}{w+1}\right)$

Numerical data. For $n = 0, \; m = 300, \; k = 1$

$$\left| \frac{\mu_{n,m,k}}{\left. \frac{d}{dw} \tilde{\zeta}(w) \right|_{w=w_{n+k}} w_{n+k}^{-m}} \right| = 0.555555555555555555555555555566973919400...$$

Numerical Conjecture A.

$$A_{0,1} = \frac{5}{9}$$

Conjecture A. For every n, k there is a rational number $A_{n,k}$ such that for $m \rightarrow \infty$

$$|\mu_{n,m,k}| = \left| (A_{n,k} + o(1)) \frac{d}{dw} \tilde{\zeta}(w) \right|_{w=w_{n+k}} w_{n+k}^{-m}$$

Numerical data. For $n = 0, m = 512, k = 2$

$$\left| \frac{\mu_{n,m,k}}{\frac{d}{dw} \tilde{\zeta}(w)} \right|_{w=w_{n+k}} w_{n+k}^{-m} = 4.41000000000000314765317394722...$$

Numerical Conjecture A.

$$A_{0,2} = \frac{441}{100} = \frac{3^2 \times 7^2}{2^2 \times 5^2}$$

Conjecture A. For every n, k there is a rational number $A_{n,k}$ such that for $m \rightarrow \infty$

$$|\mu_{n,m,k}| = \left| (A_{n,k} + o(1)) \frac{d}{dw} \tilde{\zeta}(w) \right|_{w=w_{n+k}} w_{n+k}^{-m}$$

Numerical data. For $n = 0, m = 1000, k = 3$

$$\begin{aligned} \left| \frac{\mu_{n,m,k}}{\frac{d}{dw} \tilde{\zeta}(w)} \right|_{w=w_{n+k}} w_{n+k}^{-m} &= 40.629145408163276971059161776592809... \\ &= [40, 1, 1, 1, 2, 3, 2, 1, 1, 6, 3, 8716968, 1, 3, 15, \dots] \end{aligned}$$

For $n = 0, m = 1024, k = 3$

$$\begin{aligned} \left| \frac{\mu_{n,m,k}}{\frac{d}{dw} \tilde{\zeta}(w)} \right|_{w=w_{n+k}} w_{n+k}^{-m} &= 40.629145408163270179386846358800... \\ &= [40, 1, 1, 1, 2, 3, 2, 1, 1, 6, 3, 20865457, 2, 6, 1, \dots] \end{aligned}$$

Numerical Conjecture A.

$$A_{0,3} = [40, 1, 1, 1, 2, 3, 2, 1, 1, 6, 3] = \frac{127413}{3136} = \frac{3^4 \times 11^2 \times 13}{2^6 \times 7^2}$$

Conjecture A. For every n, k there is a rational number $A_{n,k}$ such that for $m \rightarrow \infty$

$$|\mu_{n,m,k}| = \left| (A_{n,k} + o(1)) \frac{d}{dw} \tilde{\zeta}(w) \right|_{w=w_{n+k}} w_{n+k}^{-m}$$

Numerical data. For $n = 1, m = 511, k = 1$

$$\begin{aligned} \left| \frac{\mu_{n,m,k}}{\frac{d}{dw} \tilde{\zeta}(w)} \right|_{w=w_{n+k}} w_{n+k}^{-m} &= 2.739130434782610541268416612253... \\ &= [2, 1, 2, 1, 5, 1024242810887, 2, 1, 1, \dots] \end{aligned}$$

Numerical Conjecture A.

$$A_{1,1} = [2, 1, 2, 1, 5] = \frac{63}{23} = \frac{3^2 \times 7}{23}$$

Conjecture A. For every n, k there is a rational number $A_{n,k}$ such that for $m \rightarrow \infty$

$$|\mu_{n,m,k}| = \left| (A_{n,k} + o(1)) \frac{d}{dw} \tilde{\zeta}(w) \right|_{w=w_{n+k}} w_{n+k}^{-m}$$

Numerical data. For $n = 1, m = 1023, k = 2$

$$\left| \frac{\mu_{n,m,k}}{\frac{d}{dw} \tilde{\zeta}(w)} \right|_{w=w_{n+k}} w_{n+k}^{-m} = 31.502612229102170887660917374\dots$$

$$= [31, 1, 1, 95, 4, 1, 10, 2526428, 4, 1, 29, \dots]$$

Numerical Conjecture A.

$$A_{1,2} = [31, 1, 1, 95, 4, 1, 10] = \frac{325611}{10336} = \frac{3^2 \times 11^2 \times 13 \times 23}{2^5 \times 17 \times 19}$$

Conjecture A. For every n, k there is a rational number $A_{n,k}$ such that for $m \rightarrow \infty$

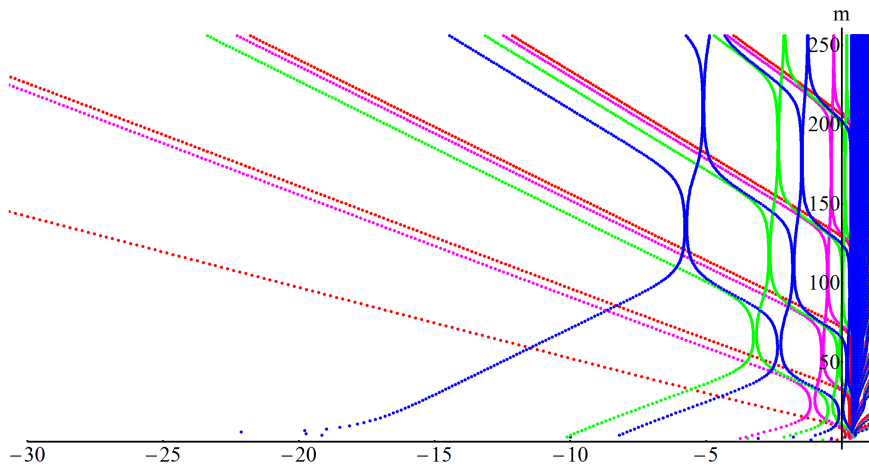
$$|\mu_{n,m,k}| = \left| (A_{n,k} + o(1)) \frac{d}{dw} \tilde{\zeta}(w) \right|_{w=w_{n+k}} w_{n+k}^{-m}$$

Numerical data. For $n = 2, m = 998, k = 1$

$$\begin{aligned} \left| \frac{\mu_{n,m,k}}{\frac{d}{dw} \tilde{\zeta}(w)} \right|_{w=w_{n+k}} w_{n+k}^{-m} &= 13.61515986029016995171114942166491261... \\ &= [13, 1, 1, 1, 1, 2, 25, 1, 16, 2, 2, 334229, 19, 5, 1, ...] \end{aligned}$$

Numerical Conjecture A.

$$A_{2,1} = [13, 1, 1, 1, 1, 2, 25, 1, 16, 2, 2] = \frac{405405}{29776} = \frac{3^4 \times 5 \times 7 \times 11 \times 13}{2^4 \times 1861}$$



Values of $\log |\mu_{n,m,k}|$, for $n = 1, 2, 3, 4$

$$|\mu_{n,m,k}| = \left| (A_{n,k} + o(1)) \frac{d}{dw} \tilde{\zeta}(w) \right|_{w=w_{n+k}} w_{n+k}^{-m}$$

RH: For all n for $m \rightarrow \infty$

Vers. 7: $|\det(M_{n,m})|^{\frac{1}{m}} \rightarrow \prod_{j=1}^n \frac{2j+1}{2j}$

Vers. 8: $(\prod_{k=1}^m |\mu_{n,m,k}|)^{\frac{1}{m}} \rightarrow \prod_{j=1}^n \frac{2j+1}{2j}$

Vers. 9: $\frac{1}{m} \sum_{k=1}^m \ln |\mu_{n,m,k}| \rightarrow \sum_{j=1}^n \ln \left(\frac{2j+1}{2j} \right)$

Vers. 10: $\frac{1}{m} \sum_{k=1}^m \ln |\mu_{n+1,m,k}| - \frac{1}{m} \sum_{k=1}^m \ln |\mu_{n,m,k}| \rightarrow \ln \left(\frac{2n+3}{2n+2} \right)$

Vers. 11: $\frac{1}{m} \sum_{k=1}^m \ln |\mu_{n+1,m,k}| - \frac{1}{m+1} \sum_{k=1}^{m+1} \ln |\mu_{n,m+1,k}| \rightarrow \ln \left(\frac{2n+3}{2n+2} \right)$

Vers. 12: $\frac{1}{m} \sum_{k=1}^m \ln |\mu_{n+1,m,k}| - \frac{1}{m} \sum_{k=1}^{m+1} \ln |\mu_{n,m+1,k}| \rightarrow \ln \left(\frac{2n+3}{2n+2} \right)$

Vers. 13:

$$\frac{1}{m} \sum_{k=1}^m (\ln |\mu_{n+1,m,k}| - \ln |\mu_{n,m+1,k+1}|) - \frac{1}{m} \ln |\mu_{n,m+1,1}| \rightarrow \ln \left(\frac{2n+3}{2n+2} \right)$$

RH (version 13)

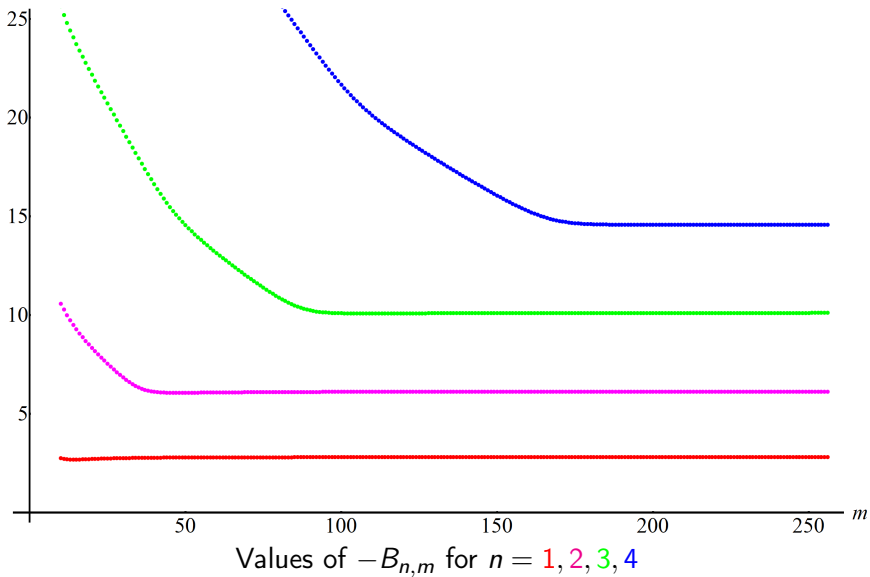
$$\frac{1}{m} \sum_{k=1}^m (\ln |\mu_{n+1,m,k}| - \ln |\mu_{n,m+1,k+1}|) - \frac{1}{m} \ln |\mu_{n,m+1,1}| \rightarrow \ln \left(\frac{2n+3}{2n+2} \right)$$

Conjecture A. $|\mu_{n,m,1}| = \left| (A_{n,1} + o(1)) \frac{d}{dw} \tilde{\zeta}(w) \Big|_{w=w_{n+1}} w_{n+1}^{-m} \right|$

Conjecture B. *For all n*

$$B_{n,m} = \sum_{k=1}^m (\ln |\mu_{n+1,m,k}| - \ln |\mu_{n,m+1,k+1}|) = o(m)$$

$$\mathbf{A} \implies (\mathbf{B} \iff \mathbf{RH})$$



Conjecture B (stronger form). *For every n there exists a number B_n such that for $m \rightarrow \infty$*

$$B_{n,m} = \sum_{k=1}^m (\ln |\mu_{n+1,m,k}| - \ln |\mu_{n,m+1,k+1}|) \rightarrow B_n.$$

Numerical data.

$$\begin{aligned} e^{B_{1,498}} &= 0.060869565217391334297063067299596... \\ &= [0, 16, 2, 2, 1, 2524751023850, 3, 20, 1, 1, 3, 11, 3, ...] \\ e^{B_{1,998}} &= [0, 16, 2, 2, 1, 2420082889472848925180620903, 3, 2, 4, 3, 5, 1, ...] \end{aligned}$$

$$[0, 16, 2, 2, 1] = \frac{7}{115} = \frac{7}{5 \times 23}$$

Numerical Conjecture B. $B_1 = \ln \left(\frac{7}{115} \right)$

Conjecture B (stronger form). *For every n there exists a number B_n such that for $m \rightarrow \infty$*

$$B_{n,m} = \sum_{k=1}^m (\ln |\mu_{n+1,m,k}| - \ln |\mu_{n,m+1,k+1}|) \rightarrow B_n.$$

Numerical data.

$$\begin{aligned} e^{B_{2,897}} &= 0.00219544023950257752579748045973779365... \\ &= [0, 455, 2, 23, 3, 39656499, 38, 14, 2, 1, 1, 3, 1, 7, , ...] \\ e^{B_{2,997}} &= 0.00219544023950257173864042772451897584... \\ &= [0, 455, 2, 23, 3, 1505709159, 2, 1, 1, 1, 3, 8, ...] \end{aligned}$$

$$[0, 455, 2, 23, 3] = \frac{143}{65135} = \frac{11 \times 13}{5 \times 7 \times 1861}$$

Numerical Conjecture B. $B_2 = \ln \left(\frac{143}{65135} \right)$

Conjecture B (stronger form). *For every n there exists a number B_n such that for $m \rightarrow \infty$*

$$B_{n,m} = \sum_{k=1}^m (\ln |\mu_{n+1,m,k}| - \ln |\mu_{n,m+1,k+1}|) \rightarrow B_n.$$

Numerical data.

$$\begin{aligned} e^{B_{3,1796}} &= 0.0000405915072581739248841180152296337... \\ &= [0, 24635, 1, 2, 3, 1, 1, 3, 2, 1082513488, 1, 2, 4, 3, 2, 3, , , ...] \\ e^{B_{3,1996}} &= 0.0000405915072581739248410776752564941286... \\ &= [0, 24635, 1, 2, 3, 1, 1, 3, 2, 96917181255, 273, 1, 1, 3, 2, , , ...] \end{aligned}$$

$$[0, 24635, 1, 2, 3, 1, 1, 3, 2] = \frac{187}{4606875} = \frac{11 \times 17}{3^4 \times 5^4 \times 7 \times 13}$$

Numerical Conjecture B. $B_3 = \ln \left(\frac{187}{4606875} \right)$

Conjecture B (stronger form). *For every n there exists a number B_n such that for $m \rightarrow \infty$*

$$B_{n,m} = \sum_{k=1}^m (\ln |\mu_{n+1,m,k}| - \ln |\mu_{n,m+1,k+1}|) \rightarrow B_n.$$

Numerical data.

$$\begin{aligned} e^{B_{4,3996}} &= .000000460297681987078674528702352178315230... \\ &= [0, 2172507, 7, 4, 1, 1, 1, 9, 1, 2, 1, 24667496266172, 2, 4, 4, \dots] \end{aligned}$$

$$\begin{aligned} [0, 2172507, 7, 4, 1, 1, 1, 9, 1, 2, 1] &= \frac{4199}{9122357475} \\ &= \frac{17 \times 19}{3^2 \times 5^2 \times 7 \times 11 \times 526543} \end{aligned}$$

Numerical Conjecture B. $B_4 = \ln \left(\frac{4199}{9122357475} \right)$

Conjecture B (stronger form). *For every n there exists a number B_n such that for $m \rightarrow \infty$*

$$B_{n,m} = \sum_{k=1}^m (\ln |\mu_{n+1,m,k}| - \ln |\mu_{n,m+1,k+1}|) \rightarrow B_n.$$

Numerical data.

$$e^{B_{5,3996}} = .0000000035346125152890138781699703763831718620\dots$$

$$\frac{185725}{52544656365201} = \frac{5^2 \times 17 \times 19 \times 23}{3^2 \times 7 \times 11 \times 13 \times 19793 \times 294673}$$

Numerical Conjecture. $B_5 = \ln \left(\frac{185725}{52544656365201} \right)$

Numerical Conjectures B.

$$B_0 = \ln \left(\frac{5}{9} \right) = \ln \left(\frac{5}{3^2} \right)$$

$$B_1 = \ln \left(\frac{7}{115} \right) = \ln \left(\frac{7}{5 \times 23} \right)$$

$$B_2 = \ln \left(\frac{143}{65135} \right) = \ln \left(\frac{11 \times 13}{5 \times 7 \times 1861} \right)$$

$$B_3 = \ln \left(\frac{187}{4606875} \right) = \ln \left(\frac{11 \times 17}{3^4 \times 5^4 \times 7 \times 13} \right)$$

$$B_4 = \ln \left(\frac{4199}{9122357475} \right) = \ln \left(\frac{17 \times 19}{3^2 \times 5^2 \times 7 \times 11 \times 526543} \right)$$

$$B_5 = \ln \left(\frac{185725}{52544656365201} \right) = \ln \left(\frac{5^2 \times 17 \times 19 \times 23}{3^2 \times 7 \times 11 \times 13 \times 19793 \times 294673} \right)$$

$$B_6 = ?$$

Numerical Conjectures B for odd characters.

$$\tilde{B}_0 = \ln \left(\frac{3}{4} \right) = \ln \left(\frac{3}{2^2} \right)$$

$$\tilde{B}_1 = \ln \left(\frac{105}{704} \right) = \ln \left(\frac{3 \times 5 \times 7}{2^6 \times 11} \right)$$

$$\tilde{B}_2 = \ln \left(\frac{385}{46848} \right) = \ln \left(\frac{5 \times 7 \times 11}{2^8 \times 3 \times 61} \right)$$

$$\tilde{B}_3 = \ln \left(\frac{45045}{220512256} \right) = \ln \left(\frac{3^2 \times 5 \times 7 \times 11 \times 13}{2^{14} \times 43 \times 313} \right)$$

$$\tilde{B}_4 = ?$$